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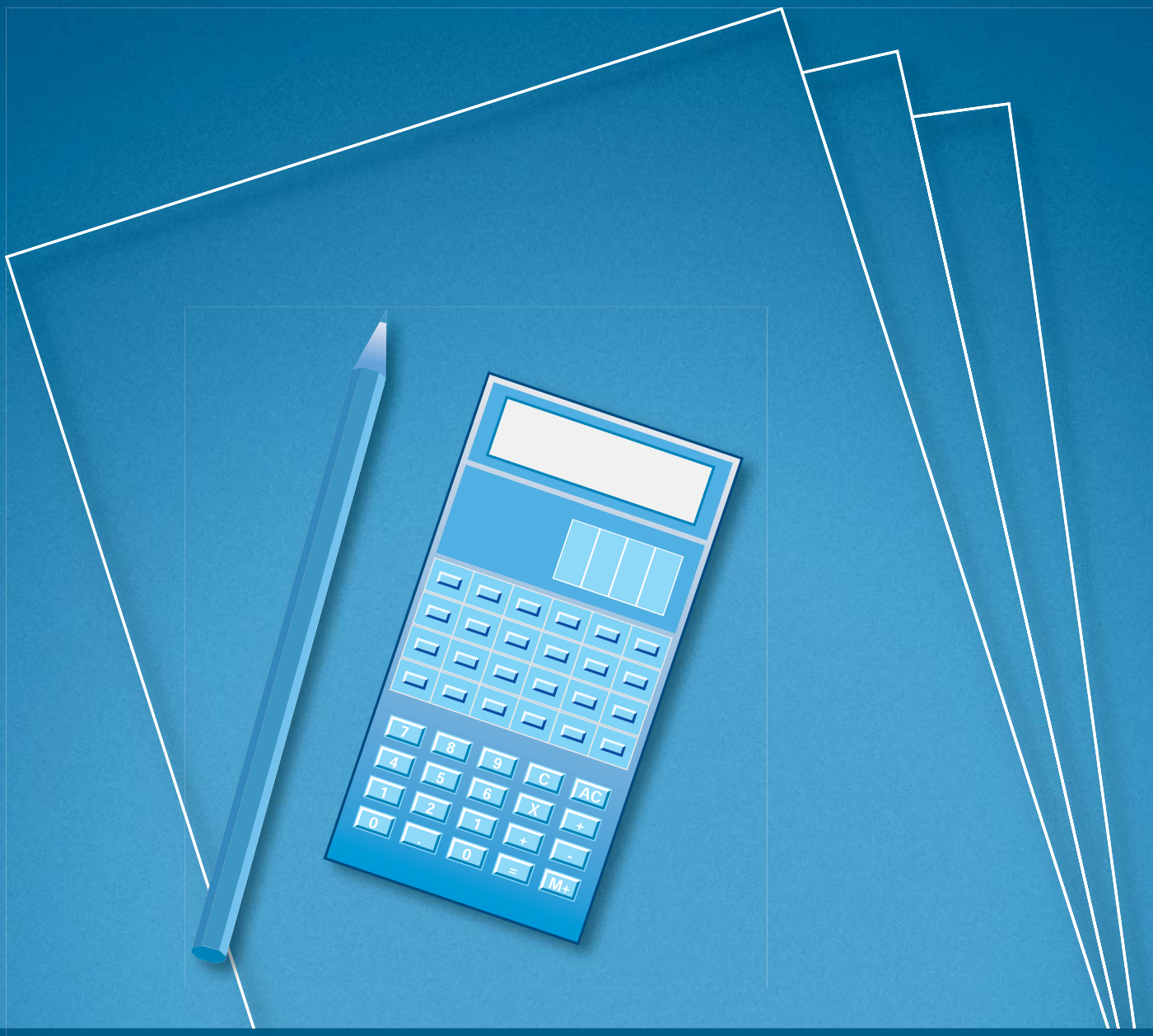

Quality Management in the Bosch Group | Technical Statistics

3. Evaluation of **Measurement Series**

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Quality Management in the Bosch Group
Technical Statistics

Booklet 3 – Evaluation of Measurement Series

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Contents

1 Introduction

Statistics are an area of mathematics whose task is that of describing, evaluating and assessing data. Their final objective is the preparation of decisions.

Statistics are generally named in the same breath as probability theory and are combined under the generic term of stochastics. This term, which comes from Greek, means roughly “the art of skillful presumption”. It expresses that the user of statistical methods attempts to select one of the theories appropriate to the situation, on the basis of limited knowledge of the facts of the case, which will allow for the best possible decision.

In doing so, the user often goes through a series of steps several times which can be summarized approximately in the following:

1. Description, representation (visualization) and illustration of experimentally determined or “observed” data.
2. The putting into concrete terms of a “presumption” about one of the “simple” mathematical models upon which the data set is based under consideration of the situation present during the data collection.
3. Inspection of the model for compatibility with the present data (observations).
4. The finding of a decision on the basis of the model postulated in 2. and not refutable in 3.

In this correlation, statistical tests are applied in order to examine whether or not a selected model is basically suitable to describe a data set. Here, it is given how large the deviation of the real data from the corresponding model data in its totality may be through the selection of an appropriate level of significance (appropriate to the effects of the decision made in step 4) in order to still be able to accept the model as adequate. Such a test occurs, naturally, only on the basis of purely mathematical points of view. Correspondingly, it can make no assessment on the extent to which it is principally at all sensible to describe the physical reality with a fictitious mathematical ideal in a concrete case.

The objective of this document is to represent several of the procedures considered significant in the correlation described above and to show the difficulties associated with them in individual cases.

2 Representation and Illustration of Data

Due to the constantly improving computer equipment and the various well-priced statistics software being offered, practically nobody today is forced to assess measurement data “by hand”. Naturally, the easy availability of efficient statistics programs is basically to be welcomed; however, they can tempt the user to consider the evaluation of measurement data as a purely statistical problem. The term “evaluation” is here all too easily limited to or even equated to the application of mathematical computation sequences. In actuality, though, “evaluation” should be understood as a result of thought processes which assess and finally lead to the further sequence within the framework of the determined measurement data under consideration of the physical/technical facts and circumstances. Therefore, the intent of explaining the “evaluation of measurement series” in this sense would mean wanting to explain human thought and the process of gaining knowledge.

The treatment of this rather philosophical topic can, naturally, not be the objective of this document. Rather, a collection of possibilities of graphic representation and self-contained statistical and general mathematical individual procedures should here be presented which support the “evaluation”, but which can never replace it. Otherwise, an expert system would be programmable which, for example, would conduct an analysis of these data at the push of a button according to the inputting of the measurement data gained within the framework of an experiment and would supply a clear report (answer), without knowledge of the facts and circumstances examined in the experiment as well as the questions upon which the experiment is based. This is obviously a self-contradiction. In Chapter 2, it should be clarified that the illustration of data is an essential and all-too-often neglected component of the evaluation of measurement series.

2.1 Original Value Diagram

Let’s consider the following table of 100 determined measurement values in temporal order and imagine that we should describe this data set in only a few words. How would one proceed? Would it be helpful to enter the values into a pocket calculator in order to calculate the average and the standard deviation?

1-10	11-20	21-30	31-40	41-50	51-60	61-70	71-80	81-90	91-100
23.85	23.82	23.83	24.08	24.08	24.06	24.04	24.30	24.34	24.25
23.90	23.88	23.85	24.12	24.09	24.08	24.04	24.33	24.30	24.31
23.89	23.86	23.92	24.16	24.14	24.08	24.12	24.07	24.38	24.33
23.96	23.94	23.88	23.92	24.15	24.13	24.12	24.15	24.37	24.31
23.95	23.93	23.95	23.92	24.16	24.17	24.16	24.14	24.40	24.42
23.99	23.93	23.94	23.96	24.20	24.17	24.17	24.10	24.16	24.40
23.97	23.96	23.97	23.96	23.95	24.21	24.22	24.20	24.19	24.44
23.79	23.98	24.05	24.07	23.95	24.28	24.22	24.23	24.20	24.37
23.79	24.01	24.00	23.98	24.02	24.24	24.26	24.26	24.23	24.25
23.80	24.05	24.07	24.05	24.02	24.03	24.24	24.29	24.26	24.24

Table 2.1

Even with longer considerations of the values in Table 2.1, it is only possible with a great deal of difficulty to find considerable qualities of this data set. On the contrary, upon looking at the following graphic representation of the values in their temporal order (original value diagram) characteristic qualities such as increase behavior, periodicity, largest and smallest values can be immediately understood.

A picture is worth more than a thousand words!

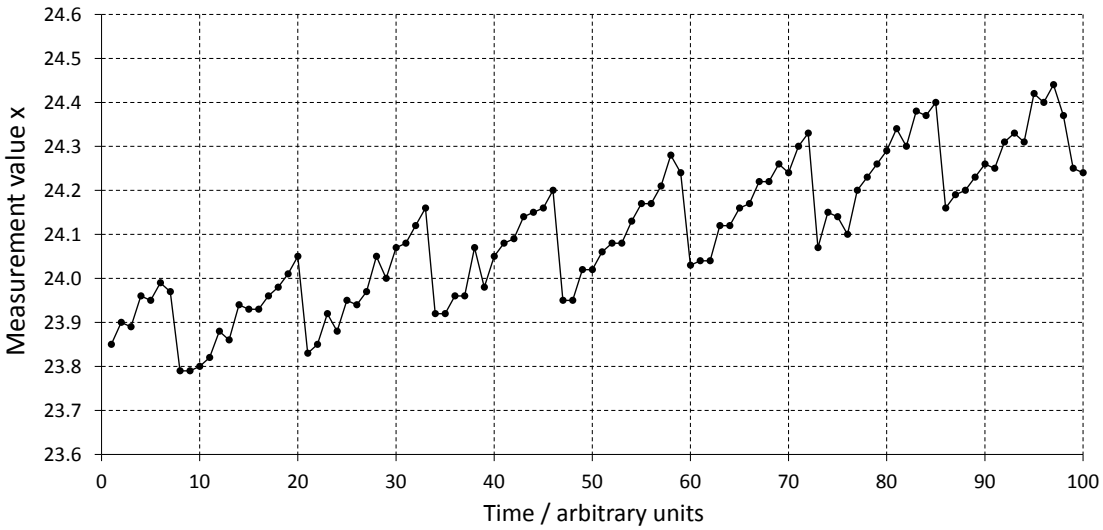


Fig. 2.1: Original value diagram of the data set from Table 2.1

2.2 Histogram

If one divides the number line (x axis) into individual areas which border one another in which one sets their borders, it is referred to as a grouping. By sorting the values of a data set into the individual classes, a number of values is produced for each class which is allotted to that class. This number is called the absolute frequency.

If one divides the absolute frequency n_j respectively by the total number n of all values, one obtains the relative frequency h_j. An application of this relative frequency to the classes in the form of rectangles lined up alongside each other is called a histogram. The histogram conveys an image of the value distribution.

For the “appearance” of the histogram, the selection of the grouping can be of decided significance. There are, however, no uniform, rigid rules for the determination of groupings, but rather merely suggestions, which are listed in the following. Statistics programs naturally require a clear, pre-set mathematical sequence for the creation of a histogram. As we will see below, the outcome of these operations does not always, however, correspond to the preconception of the user. For this reason, the programs offer, as a rule, the possibility of manually changing the number of classes and the class width. Finally, one must orient himself to the individual particulars of the present data set during the creation of a histogram.

Sequence for the manual creation of a histogram
from n individual values x_i of a data set:

1. Select a suitable grouping

   Determine the number of classes k

   Rule of thumb: $25 \le n \le 100$ $\quad k = \sqrt{n}$

   $n > 100$ $\quad k = 5 \cdot \log(n)$

   The grouping should be selected with a fixed class width (if possible) so that "simple numbers" are produced as class limits. The first class should not be open to the left and the last should not be open to the right. Empty classes, i.e. those classes into which no value from the data set falls, are to be avoided.

2. Sort the values x_i into the individual classes

   Determine the absolute frequencies n_j for $j = 1, 2, \ldots, k$

3. Calculate the relative frequencies $h_j = \frac{n_j}{n}$ for $j = 1, 2, \ldots, k$

4. Plot the relative frequencies h_j (y axis) over the individual classes of the grouping (x axis)

The suggestion of calculating the class interval b in accordance with $b = \frac{x_{max} - x_{min}}{k - 1}$ (x_{min} designates the smallest value, x_{max} the largest value of the data set), usually results in class limits with several fractional digits, which is impractical for the manual creation of a histogram and, beyond that, can lead to empty classes. Within the scope of a computer program, such a determination is, however, definitely usable, since it guarantees, in connection with an automatic, data-dependent sub-division and the marking of the coordinate axes (auto-scaling) that the histogram contains all values and that the histogram itself can be reproduced in full on the monitor. It does, though, have its dangers. An "outlier", for example, which is considerably larger than all remaining values can lead to the class interval being very large. In an unfavorable case, this can result in this value being allocated to the rightmost class and all other values ending up in the leftmost class. The classes in between then remain empty.

EXAMPLE 2.1:

We consider a sample of 42 pieces for which a tolerance $T = 0.1mm$ is given for their characteristic "length". The individual values listed in Table 2.1 are deviations of the respective length of the target value C (midpoint of the tolerance zone) in $1/100mm$. They were sorted in increasing order. The resolution of the measuring instrument used is $0.002mm$, thereby fulfilling the condition Resolution $\leq 5\% \cdot T$ within the scope of the measuring systems capability study (for currently valid value see [6]).

Deviation from target value C / $1/100mm$						
-3.8	*-2.8*	*-2.6*	*-2.4*	*-2.2*	*-2.0*	*-1.8*
-1.6	*-1.6*	*-1.4*	*-1.2*	*-1.2*	*-1.0*	*-0.8*
-0.8	*-0.8*	*-0.6*	*-0.6*	*-0.4*	*-0.2*	*-0.2*
0.0	*0.0*	*0.2*	*0.2*	*0.4*	*0.6*	*0.6*
0.8	*0.8*	*1.0*	*1.2*	*1.2*	*1.4*	*1.6*
1.6	*1.8*	*2.0*	*2.4*	*2.6*	*2.8*	*3.2*

Table 2.2

According to the rule of thumb, $k = \sqrt{42} \approx 6.48$. The number of classes should then be selected to equal either six or seven. Table 2.3 describes the groupings created by a computer program after manual input of the number of classes $k = 7$ as well as the respective absolute and relative frequency. The program calculated the class interval as the value

$$b = \frac{3.2 - (-3.8)}{7 - 1} = \frac{7}{6} \approx 1.166 .$$

Class	*1*	*2*	*3*	*4*	*5*	*6*	*7*
Lower class limit	*-4.383*	*-3.217*	*-2.050*	*-0.883*	*0.283*	*1.450*	*2.617*
Upper class limit	*-3.217*	*-2.050*	*-0.883*	*0.283*	*1.450*	*2.617*	*3.783*
Absolute frequency	*1*	*4*	*8*	*12*	*9*	*6*	*2*
Relative frequency	*2.4 %*	*9.5 %*	*19.0 %*	*28.6 %*	*21.4 %*	*14.3 %*	*4.8 %*
Cumulative Rel. frequency	*2.4 %*	*11.9 %*	*30.9 %*	*59.5 %*	*80.9 %*	*95.2 %*	*100 %*

Table 2.3

Figure 2.2 shows the associated histogram. In the retention of the number of class seven and manual determination of the class interval to 1.2 and the lower limit of the first class to -4.4, the histogram is created in accordance with Figure 2.3.

The Figures 2.4 and 2.5 display the resulting histograms for the selection of the class number 6. The adjustment of the class width is $b = 1.4$ in both cases, and merely the lower limit of the first class was changed in the given manner. In the histogram in accordance with Figure 2.5, the upper limit of the 5th class corresponds to the maximum value of Table 2.2. The sixth class thereby remains empty.

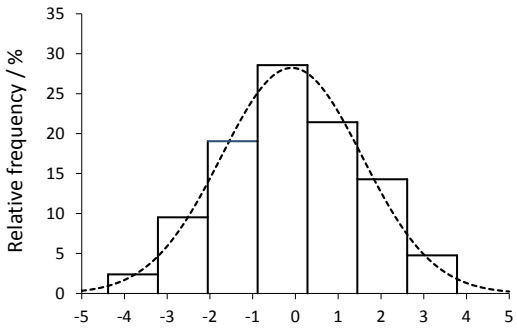


Fig. 2.2: Number of classes $k = 7$
Class width $b = 1.166$
Lower limit of the 1st class: -4.383

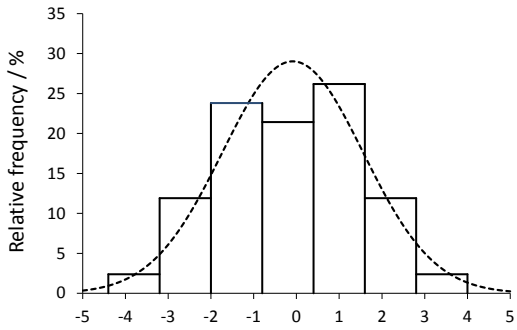


Fig. 2.3: Number of classes $k = 7$
Class width $b = 1.2$
Lower limit of the 1st class: -4.4

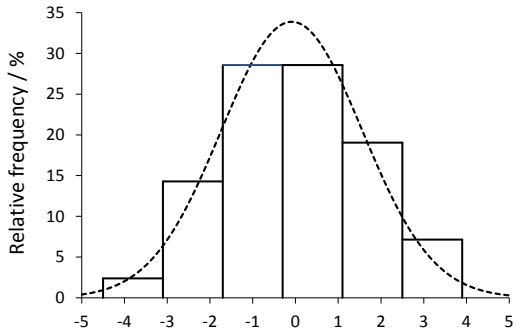


Fig. 2.4: Number of classes $k = 6$
Class width $b = 1.4$
Lower limit of the 1st class: -4.5

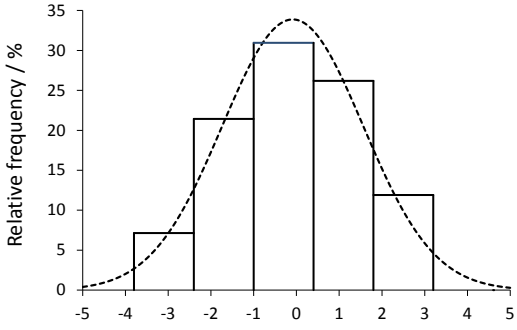


Fig. 2.5: Number of classes $k = 6$
Class width $b = 1.4$
Lower limit of the 1st class: -3.8

It is clearly a mistake to assume that the creation of a histogram is a process with a clear outcome. The examples shown are rather of harmless nature. If one selects the class width smaller than the resolution of the measuring instrument, “gaps”, i.e. empty classes, are inevitably produced in the histogram.

Finally, another particularity should be referred to in reference to the Figures 2.2 to 2.5.

The relative frequency allotted to a class from a histogram corresponds to the probability with which a value of the examined population will end up in this class. Expressed mathematically, the relative frequency to a class corresponds with an estimation of the probability with which the random variable X upon which the population is based assumes values within the class limits (of the class considered).

The sum of the relative frequencies of all classes of a histogram has the value of one (100 %). If one interprets the total area of a histogram as the probability with which the random variable X assumes values within the range covered by the histogram on the x axis, this total area must also have the value of one.

This is, however, only correct for the class width $b = 1$; otherwise, the total area corresponds to the class interval.

This results in the histogram areas and the areas under the Gaussian bell curve in Figures 2.3 and 2.4 being respectively different.

In practice, it can therefore occur that one obtains two histograms for two measurement series of the same examined facts and circumstances (e.g. internal measurement and customer measurement) which convey varying optical impressions due to differing class widths and make a direct comparison difficult or completely impossible.

The most significant aspect in the creation of a histogram is the loss of information about the original values and particularly their temporal order. In order to clarify this, we will display special value sequences with Figures 2.6, 2.7 and 2.8, which must clearly have completely different physical origins. How, then, would the associated histograms look?

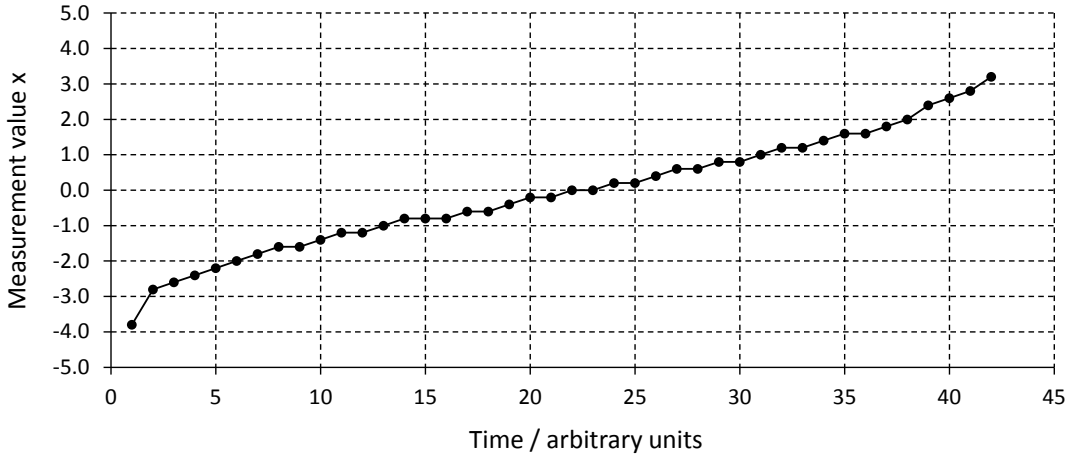


Fig. 2.6: Original value diagram for Table 2.2

The stunning thing about these Figures is the fact that the values represented therein, except for the temporal order, correspond to the data set according to Table 2.2 and all lead to the same histogram (Fig. 2.2) according to the grouping in accordance with Table 2.3. This example is admittedly constructed. It does, however, show in a remarkable way how significant the representation of an original value sequence can be if the parameter of time or the order of the values play a role in the developmental history of the data material.

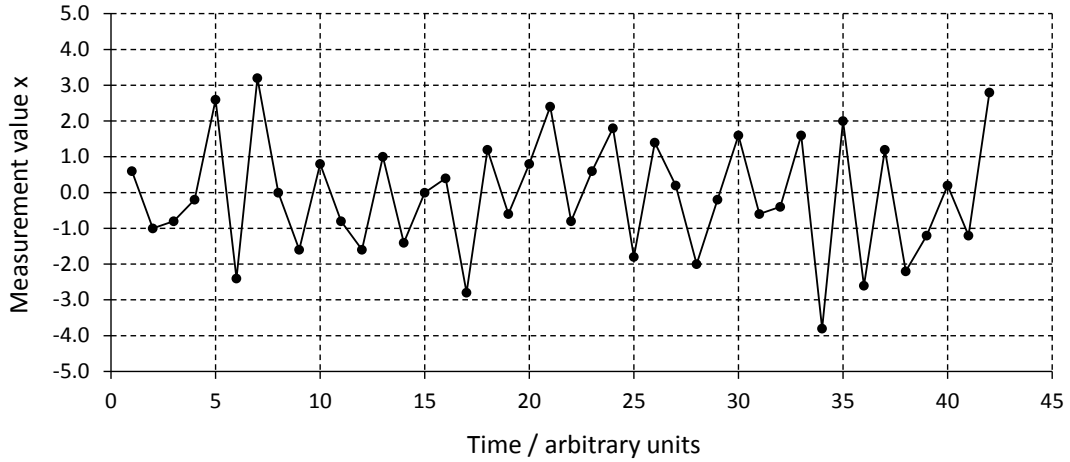


Fig. 2.7: Original value diagram for Table 2.2

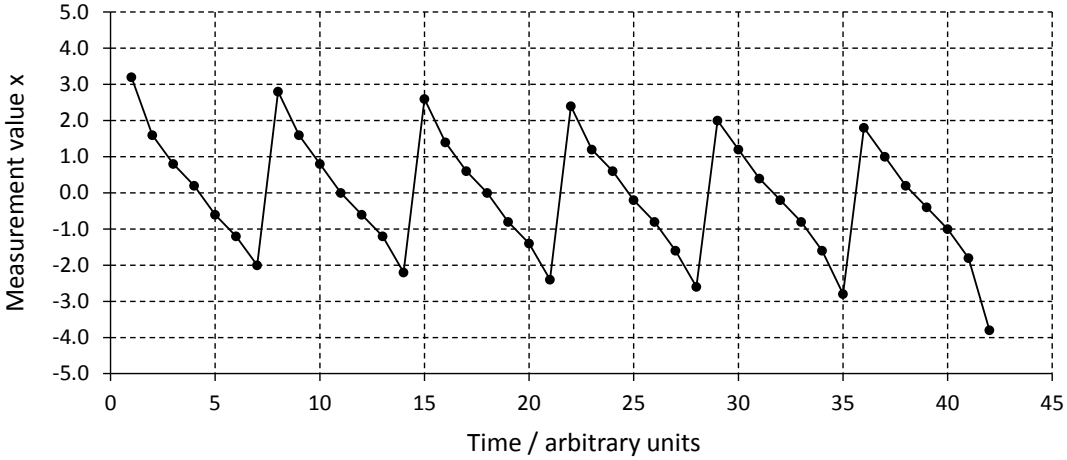


Fig. 2.8: Original value diagram for Table 2.2

2.3 Value Ray Representation and Dot Diagram

The difficulties named in correlation with the histogram can be avoided if one selects one of the following representation options. In Figures 2.9 and 2.10, the values from Table 2.2 are illustrated through the absolute frequency over the value ray, whereby they are represented as arrows in Fig. 2.9 and as circles in Fig. 2.10. Both types of representation enable, in contrast to a histogram, the reconstruction of the individual values, whereby, however, the temporal order is lost in all three cases. As long as the original values were not rounded off during their recording, the resolution of the measurement instrument can be estimated directly from the smallest lateral interval between the arrows or the dots.

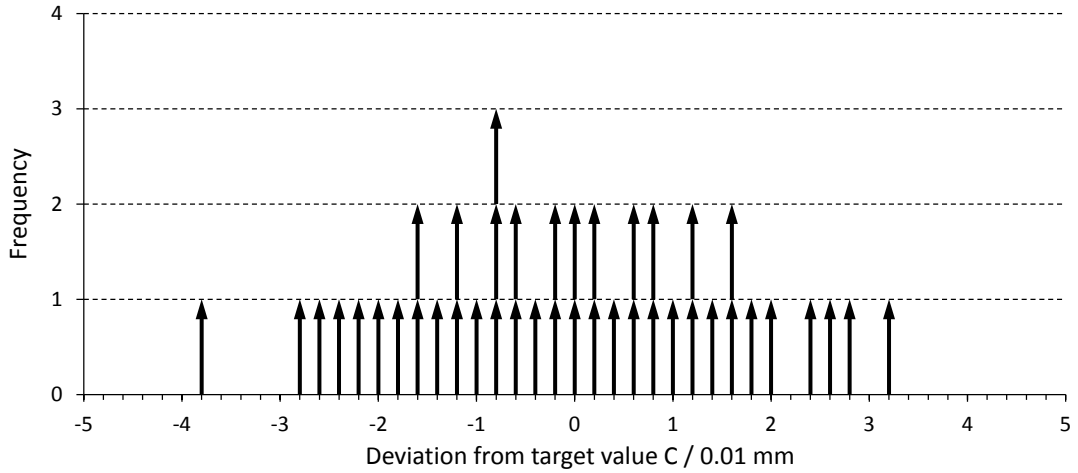


Fig. 2.9: Schematic representation of the absolute frequency of the individual values over the value ray

With the dot diagram, the scales of the x-axis and the y-axis must correspond with one another due to the symmetry of the circles (dots), since otherwise horizontal or vertical gaps will occur in the representation, or the circles will be too small or will overlap. This is a disadvantage of this type of representation versus the value ray representation according to Fig. 2.9.

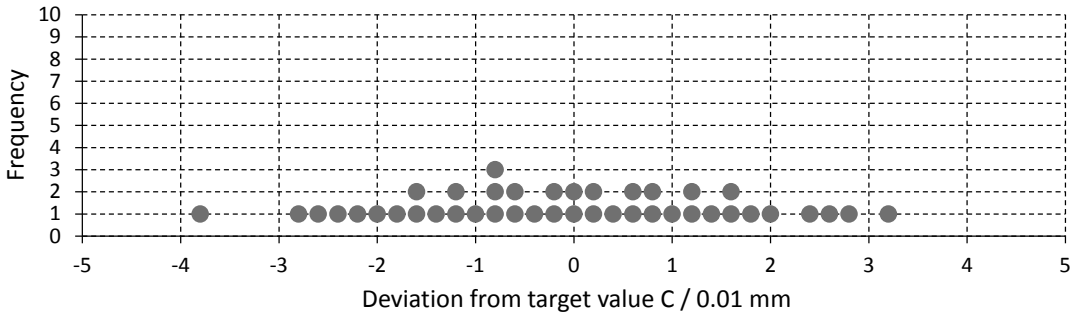


Fig. 2.10: Dot diagram of the values from Table 2.2

2.4 Stratification

The term “stratification” means basically “data separation”. It should always remind of the consideration of the developmental history of the values present during the evaluation of the data material. A suitable separation (grouping, stratification) of measured values supplies frequently valuable information on the facts being examined.

As an example, 25 averages from samples (of size $n=5$) of an $\bar{x}-s$ control chart are represented in Figures 2.11 and 2.12. With the individual values, we are dealing with outcomes of a dry inspection of magnetic valves within the scope of hydraulic modulator (ABS) production.

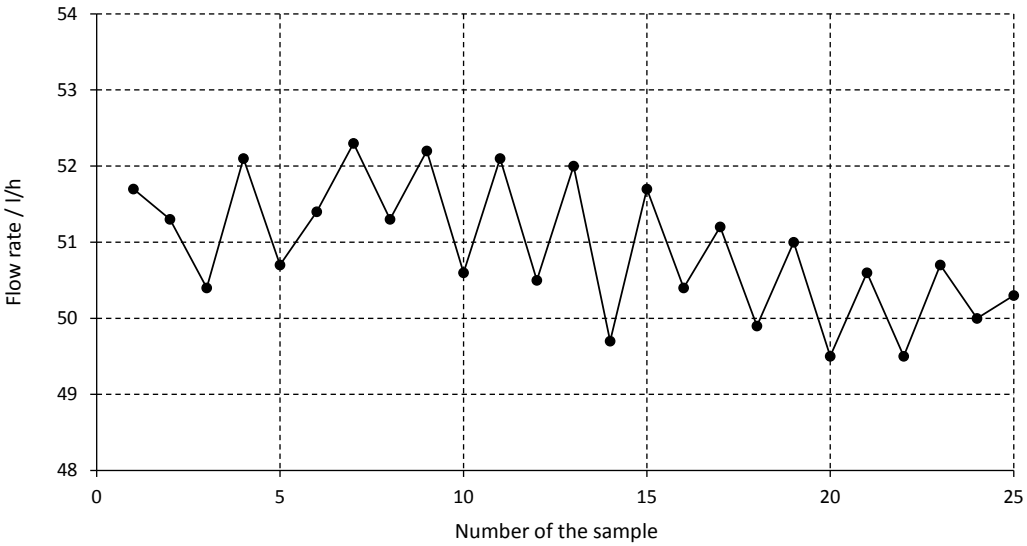


Fig. 2.11: Schematic representation of a diagram from the quality control chart

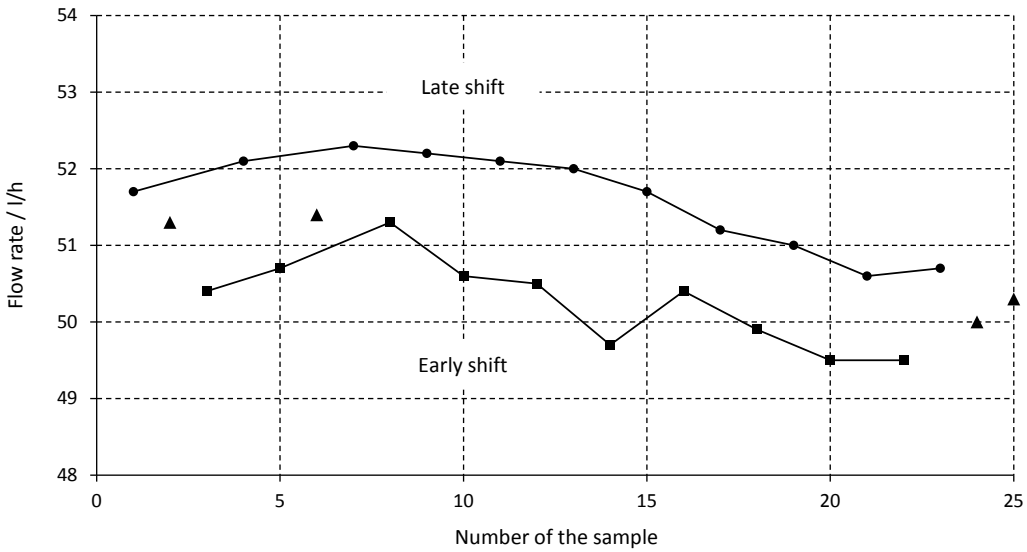


Fig. 2.12: Schematic representation of a diagram from the quality control chart (separation of the averages in correspondence with the early and late shifts)

If one considers the times which are recorded on the control chart during the sampling and connects the dots recorded respectively during the early and late shifts at approximately the same times (Fig. 2.12; the dots which were recorded at other times were clearly disregarded), a clear difference becomes obvious. Due to the temperature motion of the measurement apparatus used, the values ascertained during the late shift were always higher than those in the early shift.

Stratification is an elemental tool of quality technology (compare with [8]), which also, for example, is expressed as a method in solving problems in the multi-vari-charts recommended by Shainin (compare with [7] and [21]), among others.

What a “suitable” data separation is and under which superordinate standpoints it occurs is not a mathematical problem, but rather a question which always must be answered in a problem-oriented manner.

2.5 Box Plot

A box plot is a graphic representation for the illustration of statistical qualities of a data set. Usually, several “boxes” which belong to various data sets are represented within a representation. They enable a quick visual comparison of essential statistical characteristics of these data sets ([11], [12], [21]).

An example for such a representation is shown in the following, Fig. 2.13.

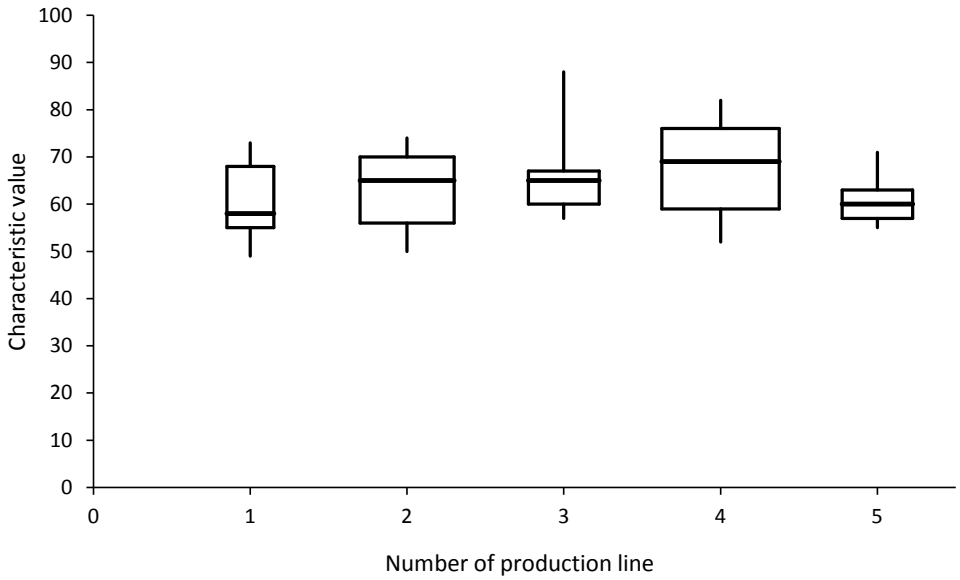


Fig. 2.13: Box plot of the characteristic values of injection valves which are manufactured on different production lines.

The individual characteristics of each box are of following significance:

- The height of the box marks a range (an interval) in which 50 % of the values of the data set lie.
- The width of the box marks the size of the accompanying sample relative to the remaining sample sizes (relative size of the data set).
- The line drawn within the box corresponds to the median (frequently, the arithmetic mean is additionally drawn in as a perforated line).
- The vertical lines above and below the box connect it with both extreme values (largest value and smallest value).

In the literature and during the realization of the box plot in statistics programs, variants are frequently found which deviate from these determinations. In particular, the boxes can also be represented horizontally over the x-axis as the characteristic axis.

The lines going out from the box are called “whiskers”, and this type of representation is therefore also called the box and whisker plot.

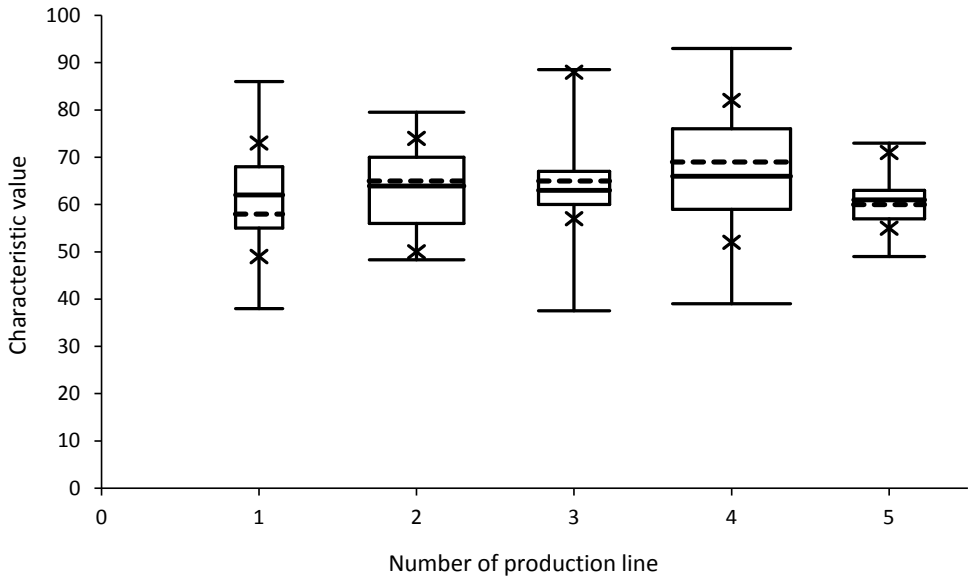


Fig. 2.14: Box plot of the data sets upon which Fig. 2.13 is also based. A comparison of the figures shows that the visual impression can be varied despite having the same data basis and depends on the conventions according to which the representation occurs.

The individual characteristics of the representation are of following significance in this case:

- The height of the box marks a range (an interval) in which 50 % of the values of the population lie, as long as it is normally distributed.
- The width of the box marks the size of the accompanying sample relative to the remaining sample sizes.
- The line drawn within the box corresponds to the arithmetic mean, and the perforated line corresponds to the median.
- The vertical lines mark the $\bar{x} \pm 3s$ range.
- The stars on the vertical lines above and below the box mark both extreme values (largest and smallest values of the sample).

Additional Representation Options:

Should one wish to compare sample outcomes with various specified characteristics, it is helpful if the tolerances or the specification limits are represented in a normalized manner. The upper limiting value then corresponds to the number +1 and the lower limiting value corresponds to the number -1. With such a representation, the respective position in the tolerance zone and the relative tolerance exploitation can be assessed quite well.

Some statistics programs offer the option of additionally marking confidence intervals of the average or the median within the box in the representation according to Fig. 2.14 through (smaller) frames.

3 Modelling

The methods of the descriptive statistics which are represented in Section 2 are used in order to make the data gained more accessible, so to say, to human examination. The wish that stands in the foreground here is that of preparing the information contained in the individual values so that they can be described with few words or with a smaller amount of data.

This is not necessarily a consciously controlled process. If one records a value sequence in correspondence with the temporal order of its occurrence as dots in an original value diagram, then one is always unconsciously attempting to uncover characteristic qualities (maximum, minimum, starting point, end point, approximate “middle”) or simple structures of this image. It is here dependent on the “appearance” of the image, to which detail attention is directed and which structure is recognized.

A data set with nearly identical values supplies a representation of dots whose intervals from dot to dot are comparably small (of course, the scaling plays a role here), through which the mental eye is forced to place a compensating line through the dots.

The same applies for “non-linear” dot sequences. The spatial proximity of adjacent dots also forces the approximation here through “smooth” curves. Recognizable regularities (constancy, tendency, periodicity) are therefore often spoken of. An unconscious simplification can already be found here, in which the eye can hardly disconcert through few dots which do not fit into this structure (“outliers”). If their number increases, however, i.e. if the intervals (temporal) between the adjacent dots increases, it will generally also become more difficult to detect a structure in the dot sequence and one will tend to speak of a chaotic (random) “course” or “behavior”.

The transition to the representational type of “histogram” is a step in abstraction, by which it is attempted, under declination of a portion of the original information (the individual values and their temporal order), to find an order on the basis of the absolute or relative frequencies within a (strictly speaking) arbitrary grouping.

One also orientates himself here to the “appearance” of the histogram — for example, its width, gaps (empty classes), distinctions in the bar height, symmetry — and attempts to find a “recognizable regularity”, i.e. allocating a simple two-dimensional structure (rectangle, triangle, trapezoid) to the formation of the adjacently bordering vertical bars in their entirety. This corresponds to an inductive manner of thinking.

With a graphic evaluation, then, one attempts, at last, to remove himself from the individual measured values and to draw conclusions on the unknown and in many cases (example: machine and process capability study) not really existent population.

3.1 Smoothing of Dot Sequences

The tendency for the human “eye” to smooth dot sequences which was addressed above is offered assistance by way of a procedure described in this section. This is a first step in the direction of a frequently striven for data description via simple mathematical functions. The procedure consists simply of replacing each dot in a dot sequence with an average, which is calculated from its $2 \cdot m$ adjacent dots and itself. The k^{th} value x_k^{old} is then replaced by

$$x_k^{new} = \frac{1}{2 \cdot m + 1} \cdot \sum_{i=-m}^{m} x_{k+i}^{old} = \frac{x_{k-m}^{old} + x_{k-m+1}^{old} + \ldots + x_{k+m-1}^{old} + x_{k+m}^{old}}{2 \cdot m + 1}.$$

Since the first m values of a sequence of n numbers have less than m adjacent dots to the left, these first m values are left unchanged during the smoothing. The same applies for the last m values of the number sequence. For the index k, the following is therefore valid: $k = m+1, m+2, \ldots, n-m$.

The manner of functioning of such a smoothing should be clarified by Fig. 3.1.

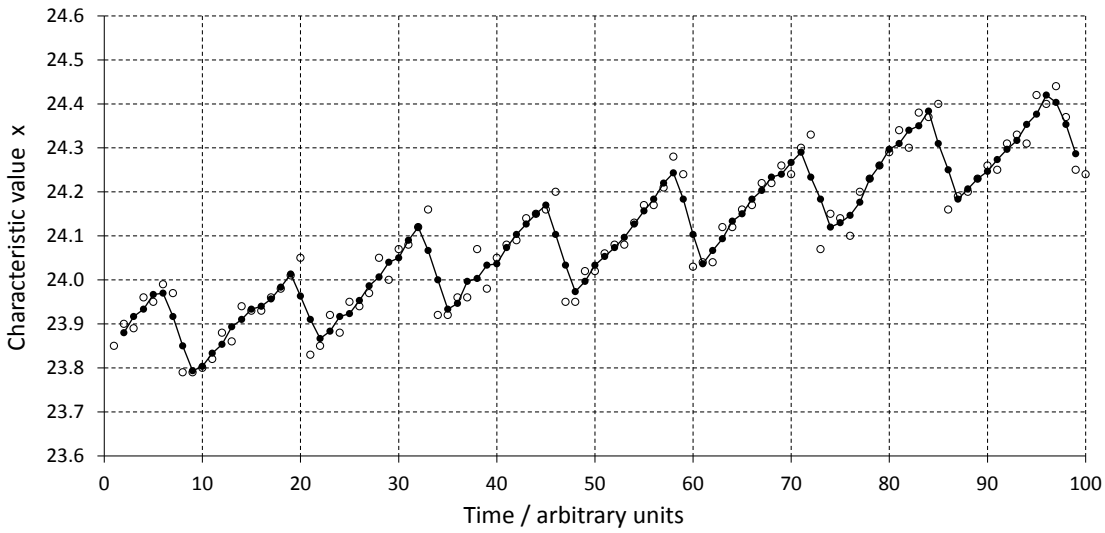


Fig. 3.1: Outcome of a smoothing algorithm. Hollow circles: Original measured values; Filled circles: “Measured values” according to the smoothing

The hollow circles correspond to the dots in the representation of Fig. 2.1. The accompanying values from Table 2.1 were handled in correspondence to the above rule with $m = 1$, i.e. the k^{th} value x_k^{old} was replaced by $x_k^{new} = \frac{x_{k-1}^{old} + x_k^{old} + x_{k+1}^{old}}{3}$, with $k = 2, 3, \ldots, 99$.

There are several variants for both the handling of the starting and end values of the number sequence as well as for the algorithm itself. For example, weighting factors could be used in the formation of the average which would weaken the effect of the kth value on dots which lie “farther away”. The procedure resembles the moving average formation, in which, however, merely the previous m values can be taken into consideration. The “filter effect” of the moving average formation is occasionally used in order to create a stable indication with continuously occurring values of a measurement signal afflicted with errors.

From Fig. 3.1, it is clear that the desired smoothing of the dot sequence is, in fact, achieved, but this leads, on the other hand, to a modification which can lead to false con-

clusions. For example, the increase of the right flank of a “tooth” as opposed to the original increase is clearly reduced (compare with Fig. 2.1)

It can thereby be determined that through the smoothing the original information contained in the data is falsified. Moreover, it can never be founded as to why the “smoother” curve should be the “more correct”, i.e. the curve which better describes reality.

3.2 Linear Interpolation

The linear interpolation is a useable procedure in determining intermediate values of a tabulated function, for example. Here, the bent curve course of the function $y = f(x)$ between two base points (explanation of term in Section 3.5) x_1 and x_2 approximates a part of a line, which runs through the dots $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

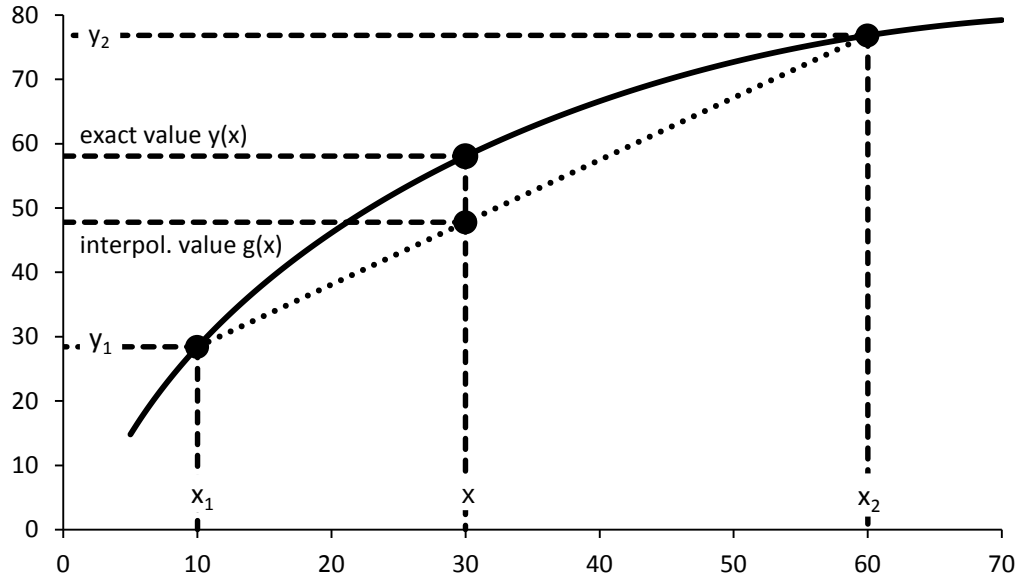


Fig. 3.2: Linear interpolation

From the relation $\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$ the equation for the linear interpolation is produced through resolution according to y: $y = g(x) = y_1 + \frac{y_2 - y_1}{x_2 - x_1} \cdot (x - x_1)$.

If one enters the interested value x with $x_1 \le x \le x_2$ into this linear equation, one will obtain the intermediate value sought after.

EXAMPLE 3.1:

With the assistance of a table of the standard normal distribution, a quantile u can be determined for the probability $P = \Phi(u) = 99\,\%$. The values u are given only in steps of 0.1 in the table:

$u = 2.1 \qquad \Phi(u) = 0.9821$

$u = 2.2 \qquad \Phi(u) = 0.9861$

$u = 2.3 \qquad \Phi(u) = 0.9893$

$u = 2.4 \qquad \Phi(u) = 0.9918$

$u = 2.5 \qquad \Phi(u) = 0.9938$

The sought after value u clearly lies between 2.3 and 2.4.

If one enters the pairs $x_1 = 0.9893$, $y_1 = 2.3$ and $x_2 = 0.9918$, $y_2 = 2.4$ as well as $x = 0.99$ into the above equation, one obtains the desired value:

$$u = g(0.99) = 2.3 + \frac{2.4 - 2.3}{0.9918 - 0.9893} \cdot (0.99 - 0.9893) = 2.328.$$

As is clear from Fig. 3.2, it is a prerequisite in the application of linear interpolation that the considered function behaves somewhat "properly" in the range between the base points x_1 and x_2, i.e. that the interpolation error $y(x) - g(x)$ is adequately small.

Such an estimation is generally problem-free if one is dealing with a known function, as in the example, and the interval of the base points $x_2 - x_1$ is small enough. It is clear from the selected formulations that the giving of concrete prerequisites for the applicability of the linear interpolation is not possible for general cases. It must be decided in individual cases as to what "proper behavior" or "adequately small interpolation error" respectively indicates.

In textbooks on design of experiments, procedures are described in which several factors (parameters) are varied on two respective levels on the basis of pre-given experiment instructions (orthogonal arrays). From the measurement results obtained, a so-called response function (described by a multi-linear form) can be calculated in a purely formal manner from which one assumes that it properly describes the examined physical facts in first proximity. Without additional experiments with intermediate positions of the experiment parameters, however, it cannot be examined as to whether this assumption is justified or not (compare with [7]).

If one concentrates in this case on one of the examined parameters, the problem thereby present corresponds to the situation according to Fig. 3.2, in which both values y_1 and y_2 of the unknown function $y = f(x)$ at the base points x_1 and x_2 are estimated on the basis of the experiment results. The calculation of theoretical intermediate values of the response function through linear interpolation holds the danger of false interpretations if the "linear correlation" model describes the reality only inadequately or completely falsely.

The calculation of intermediate values is particularly entirely nonsensical if one is dealing with values of a discrete parameter (for example, supplier A, supplier B, machine no. 1, machine no. 2) with the base points x_1 and x_2.

3.3 Linear Regression

In practice, often the problem occurs of adjusting a line to the points accompanying the n pairs $(x_i;\ y_i)$ in an x-y diagram. For the following considerations, it is insignificant whether the basic physical facts is known to be described by a linear function or whether simply the "form" of the "dot cluster" produced allows for the sensible appearance of the approximation through a "best fitting" straight line (regression line).

The starting point for the solution of this problem is the demand that the sum of the square deviations at the positions x_i is minimal:

$$\sum_{i=1}^{n} \left(y_i - g_i\right)^2 = \sum_{i=1}^{n} \left(y_i - (a + b \cdot x_i)\right)^2 = \text{Min.}$$

The procedure is therefore also called the method of least squares. The slope b and the intercept a of the fitted least squares line $y = g(x) = a + b \cdot x$ should be calculated.

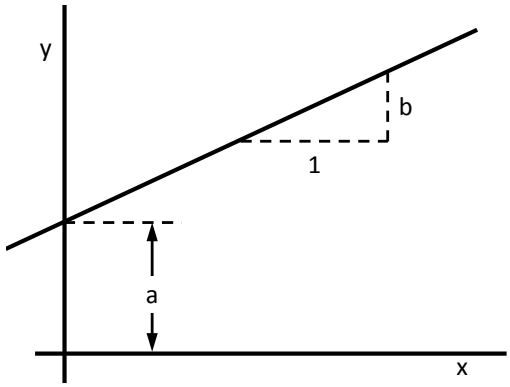


Fig. 3.3

The following formulas for a and b are produced if one partially derives the above sum of squares respectively according to a and b and determines the zero positions of the derivatives:

$$b = \frac{\frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2} = \frac{s_{xy}}{s_x^2} \qquad a = \bar{y} - b \cdot \bar{x}$$

$\bar{x}$ and s_x^2 designate therein the mean $\bar{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$ and the variance of the x_i.

$\bar{y} = \frac{1}{n} \cdot \sum_{i=1}^{n} y_i$ is the average of the y_i. The quantity b is also called the regression coefficient. The expression s_{xy} is called the covariance and can be calculated with the formula

$s_{xy} = \frac{1}{n-1} \cdot \left(\sum_{i=1}^{n} x_i \cdot y_i - n \cdot \bar{x} \cdot \bar{y} \right)$ (compare with correlation).

If one solves the expression for the intercept $a = \bar{y} - b \cdot \bar{x}$ according to $\bar{y}$, it becomes clear that the best fitting line runs through the center $(\bar{x};\ \bar{y})$ of the dot cluster: $\bar{y} = a + b \cdot \bar{x}$.

As already mentioned, a and b are determined in such a manner that the sum of the vertical square deviations of the measured values y_i from the values $g_i = g(x_i)$ given by way of the compensating line becomes minimal. There are, however, situations in which it is unclear as to whether the first or second respective value of a pair should be allocated to the x-axis. Then, both a recording of the points $(x_i;\ y_i)$ and of the points $(y_i;\ x_i)$ is possible. For both of these cases, two different least squares lines are generally produced which intersect below a specific angle at the point $(\bar{x};\ \bar{y})$. This angle becomes smaller the less the points "scatter around the fitted line".

The "quality" of the assumed or known correlation between the variables Y and X can be assessed with the assistance of the correlation coefficient r.

$$r = \frac{\frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x}) \cdot (y_i - \bar{y})}{s_x \cdot s_y} = \frac{s_{xy}}{s_x \cdot s_y}$$

$\bar{x}$ and $\bar{y}$ designate the means, s_x and s_y designate the standard deviations of x_i or of y_i. s_{xy} is the covariance of X and Y.

r can assume values between -1 and +1. A value in the proximity of +1 (-1), e.g. 0.9 (-0.9) corresponds to a dot cluster which can be approximated very well with a best fitting line with positive (negative) slope; one then refers to a strong positive (negative) correlation.

A strong correlation does not inevitably signify that Y is directly dependent upon X; it can be produced, through the dependency of both quantities X and Y, from a third quantity Z (illusory correlation).

If an exponential correlation of the form $z = v \cdot u^w$ is presumed between a quantity Z and a quantity U, the constants v and w can also be calculated with the assistance of the least-squares method. If this equation is done logarithmically, the following linear equation is produced: $\log(z) = \log(v) + w \cdot \log(u)$.

After the renaming of the individual terms, this equation assumes the form $y = a + b \cdot x$ used in the above: $\log(z) = y$, $\log(v) = a$, $\log(u) = x$, $w = b$.

If one determines, then, the constants a and b with the assistance of the pairs $(x_i = \log(u_i);\ y_i = \log(z_i))$, v can easily be calculated: $v = 10^a$. w is identical to b.

EXAMPLE 3.2:

At an air-mass flow sensor, the signal voltage U was measured as dependent on the air mass m flowing through a cross-section. The 8 measured values present should be approximated through a characteristic function in the form $U = v \cdot m^w$.

i	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*
m_i / kg/h	*15*	*30*	*60*	*120*	*280*	*370*	*480*	*640*
U_i / V	*2.2782*	*2.5531*	*2.8835*	*3.2739*	*3.9796*	*4.2574*	*4.5030*	*4.8251*

Finding the logarithms of the 8 pairs (U_i, m_i) *and the recording of the pairs* $(\log(U_i), \log(m_i))$ *in a coordinate system with linear scaling supplies the representation shown in Fig. 3.4. The regression line is drawn in.*

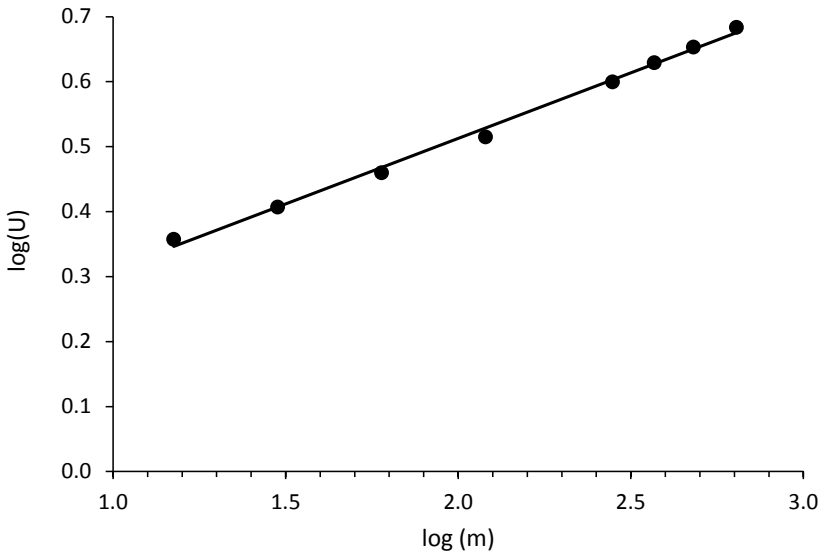


Fig. 3.4: Linear regression with the assistance of transformed measured values

The following values are produced for the intercept and the regression coefficient:

$a = 0.109$ *and* $b = 0.202$.

The value v *is found by way of reconversion:* $v = 10^{a} = 10^{0.109} = 1.285$.

With $w = b = 0.202$, $U = 1.285 \cdot m^{0.202}$ *is finally produced.*

In Fig. 3.5, the original measured values as well as the graph of the regression function U *are drawn in (solid line).*

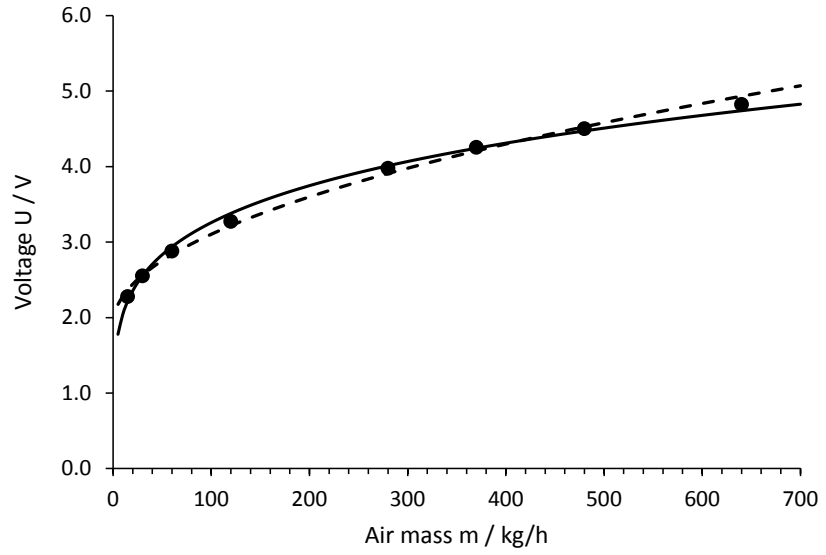


Fig. 3.5: Representation of the original values and two approximation curves

The perforated curve also drawn in Fig. 3.5 is the graph of the function $U = 1.91 + 0.29 \cdot \sqrt{\frac{m}{5.9}}$.

Clearly, the measurement points were approximated quite well by both curves. It is, of course, not possible to decide which of the two function formulations is the "better" or "right" one through the assessment of the "quality" of the approximation (for example, through comparison of the accompanying sums of squares) if nothing on the physical effects in the examined sensor and their mathematical description is known.

Naturally, another problem arises in that different dot sequences and thereby different parameter values of the approximation functions can be produced again and again during repeated measurements on the same sensor or during comparative measurements on different sensors of the same type.

3.4 Quadratic Regression

If it is known during an experimental examination that no linear relations can be present, it would be nonsensical to wish to describe the corresponding facts by way of a linear equation. If one knows, however, that the assumption of a quadratic function is sensible, or if this is suggested through the form of the dot sequence in an x-y diagram, then this leads to the problem of determining the coefficients a, b_1 and b_2 of the regression function $y = a + b_1 \cdot x + b_2 \cdot x^2$. This is also possible with the assistance of the least-squares method.

As in the linear case, the starting point for the solution of the current problem is the demand that the sum of the squared deviations at the positions x_i become minimal:

$$\sum_{i=1}^{n} \left(y_i - (a + b_1 \cdot x_i + b_2 \cdot x_i^2) \right)^2 = \text{Min.}$$

After calculation of the auxiliary quantity

$$K = n \cdot \left(\sum x_i^2\right) \cdot \left(\sum x_i^4\right) + 2 \cdot \left(\sum x_i\right) \cdot \left(\sum x_i^2\right) \cdot \left(\sum x_i^3\right) - \left(\sum x_i^2\right)^3 - n \cdot \left(\sum x_i^3\right)^2 - \left(\sum x_i\right)^2 \cdot \left(\sum x_i^4\right)$$

as well as the expressions

$$A = \left(\left(\sum x_i^2\right) \cdot \left(\sum x_i^4\right) - \left(\sum x_i^3\right)^2 \right) \cdot \frac{1}{K} \qquad B = \left(n \cdot \sum x_i^4 - \left(\sum x_i^2\right)^2 \right) \cdot \frac{1}{K}$$

$$C = \left(n \cdot \sum x_i^2 - \left(\sum x_i\right)^2 \right) \cdot \frac{1}{K} \qquad D = \left(\left(\sum x_i^3\right) \cdot \left(\sum x_i^2\right) - \left(\sum x_i\right) \cdot \left(\sum x_i^4\right) \right) \cdot \frac{1}{K}$$

$$E = \left(\left(\sum x_i\right) \cdot \left(\sum x_i^3\right) - \left(\sum x_i^2\right)^2 \right) \cdot \frac{1}{K} \qquad G = \left(\left(\sum x_i\right) \cdot \left(\sum x_i^2\right) - n \cdot \sum x_i^3 \right) \cdot \frac{1}{K}$$

the sought after coefficients of the regression parabola are produced:

$$a = A \cdot \sum y_i + D \cdot \sum x_i \cdot y_i + E \cdot \sum x_i^2 \cdot y_i$$

$$b_1 = D \cdot \sum y_i + B \cdot \sum x_i \cdot y_i + G \cdot \sum x_i^2 \cdot y_i$$

$$b_2 = E \cdot \sum y_i + G \cdot \sum x_i \cdot y_i + C \cdot \sum x_i^2 \cdot y_i \,.$$

The summations extend in all expressions over $i = 1, \ldots, n$.

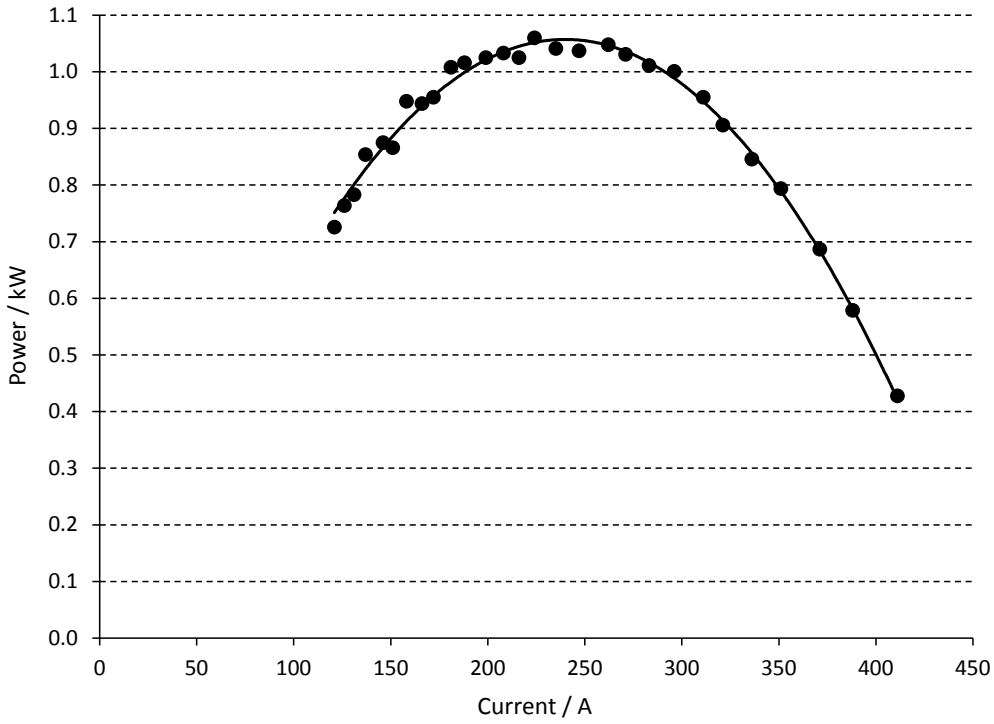


Fig. 3.6: Quadratic regression

Fig. 3.6 shows an application example of this procedure. In the representation, the power of a starter is applied over the current. This power is the product of torque and speed (number of revolutions). The speed reduces in an approximate linear manner (negative slope) depending on current, and the torque conversely increases in an approximate linear manner with the current (positive slope). The product of torque and speed is therefore inevitably, in an approximate manner, a parabola which curves downwards.

The selection of the model in the above example can be based, then, on the knowledge of the physical correlation.

3.5 Interpolation by Polynomials

If one knows the functional values $y_0 = f(x_0)$, $y_1 = f(x_1)$, …, $y_n = f(x_n)$ of a function $y = f(x)$ for the arguments $x = x_0$, $x = x_1$, …, $x = x_n$, then one also calls the latter base positions, and the functional values are then called base values.

In the procedures of the linear or square regression represented in Sections 3.3 and 3.4, one attempts to determine an approximation function whose values most closely approximate the base values at the base positions. It is taken for granted that due to the "simplicity" of the selected class of functions, only at a few points — if at all — can an exact agreement of the approximation functional values and base values be achieved.

If such an agreement is desired at all base positions, one must use correspondingly more complex functions for the modelling. One of the functional classes frequently selected for this purpose is that of polynomials, expressions of the form

$$P_n(x) = a_0 + a_1 \cdot x + a_2 \cdot x^2 + \ldots + a_n \cdot x^n \qquad n \text{ integral, } n \geq 0$$

Their selection appears favorable at first glance, since various mathematical operations such as differentiation and integration can be applied problem-free in this case. Disadvantageous in a formulation of this type is, however, the comparably great calculation

expenditure, particularly with a large quantity n (degree of the polynomial) which is necessary for the determination of the coefficients. Moreover, considerable deviations $P_n(x) - f(x)$ can be produced in the ranges between the base positions, which can be shown through the application of corresponding approximation procedures to a known function $y = f(x)$.

It is to be observed that the functions selected in 3.3 and 3.4 also correspond to such a formulation.

In this relation, considerably satisfactory outcomes can be achieved if one uses the so-called cubic spline functions for interpolation. They are combined individually from third degree polynomials ($n = 3$) in such a manner that they agree exactly with the measurement values at the base positions and, moreover, supply a “smooth” curve

As an example, Fig. 3.7 shows a correspondingly “smooth” curve course through the points from Fig. 3.6.

In no way do we wish to describe the mathematical procedure for curve approximation in detail in this third section. The corresponding calculations are generally only manageable in an acceptable amount of time by way of a computer program, anyway. Moreover, it should become clear that one can, principally, approximate or even exactly describe any dot sequences through the application of numeric interpolation procedures. The data analysis cannot, however, supply information on which of the selected models is the best or the “right” one.

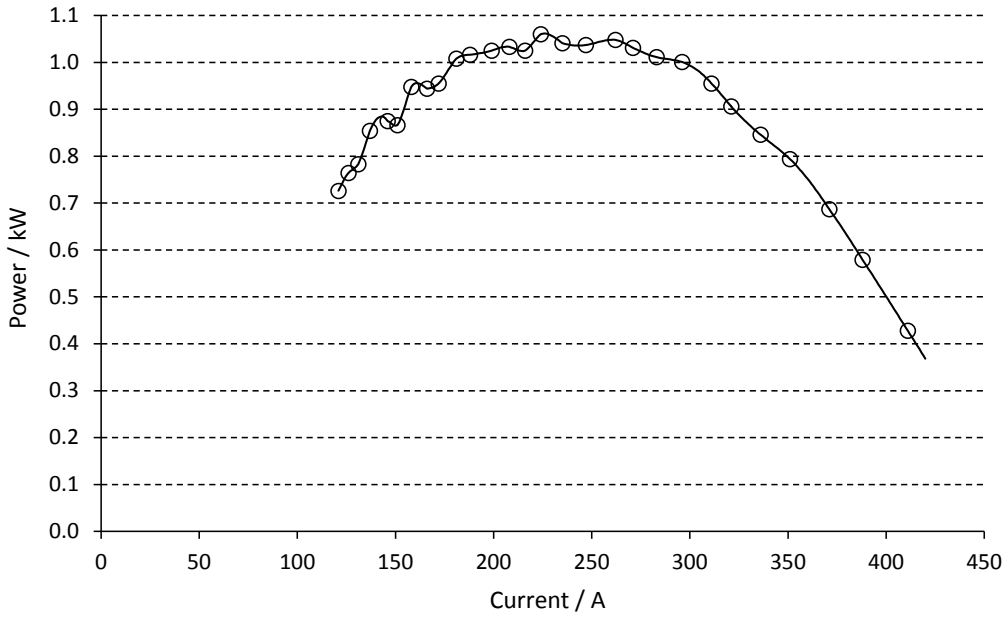


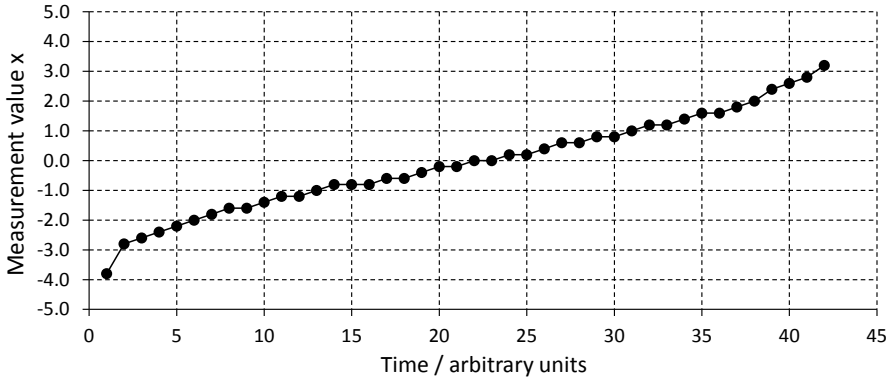
Fig. 3.7: Interpolation through a spline function (compare with Fig. 3.6)

3.6 Statistical Modelling

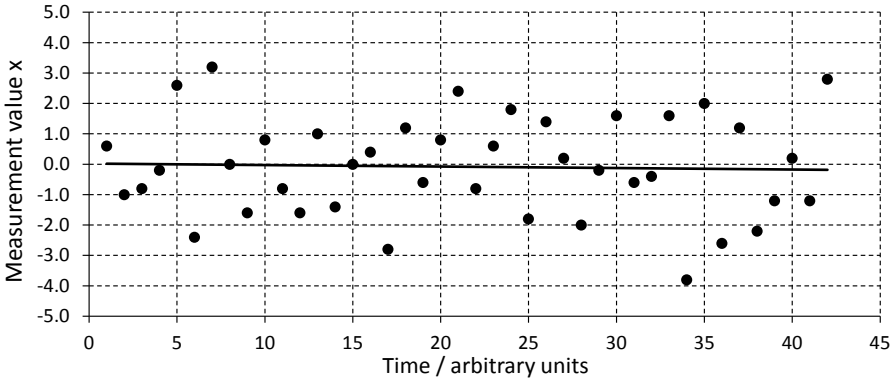
In Sections 3.3 and 3.4, procedures were explained with which measurement data were allocated to a mathematical model in the form of a “simple” function. On the basis of the Figures 3.4, 3.5 and 3.6, it becomes clear that one accepts slight deviations of the measurement points from the approximation curve. This occurs from the experience that (aside from measurement errors), individual measurement values can practically never be exactly reproduced, even under apparently unchanged conditions under which the measurement data were gained. A random component, then, comes into play here which is the subject of a statistical modelling.

Let's consider again the Figures 2.6, 2.7 and 2.8 in Section 2.2. In all three cases it is possible to describe the measurement points by way of a function $f(t)$ which is dependent on the time t. Through the dot sequence from Fig. 2.6, one can put down a compensating line with positive slope (Fig. 3.8a), in Fig. 2.8 one can select a periodic, individually linear function for the approximation (Fig. 3.8c) and in the case of Fig. 2.7, a line parallel to the time axis is “fitting” (Fig. 3.8b).

a)



b)



c)

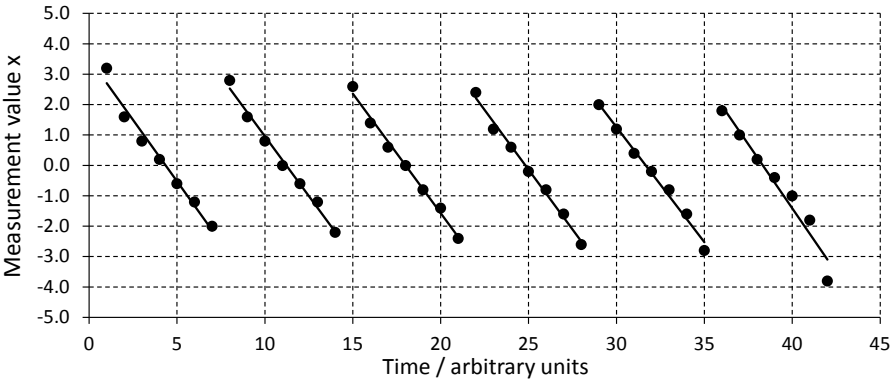


Fig. 3.8: Dot sequences from the Figures 2.6, 2.7 and 2.8 with compensating lines

In the cases a) and c), the deviation of the individual points from the compensating lines is comparatively small, while in case b) considerable deviations are produced. In all three cases, however, these deviations clearly show no systematic behavior, but rather appear to be of random nature.

It is, then, conceivable to describe the measurement values x through a function which contains a temporally dependent $f(t)$ as well as a random component X:

$$x = f(t) + X$$

X is a so-called random variable (compare with Section 4.). In the situation according to Fig. 3.8b), $f(t) = 0$; the measurement values x are then values of the random variable X.

The histogram from Fig. 2.2 gives an impression of the distribution of this random variable.

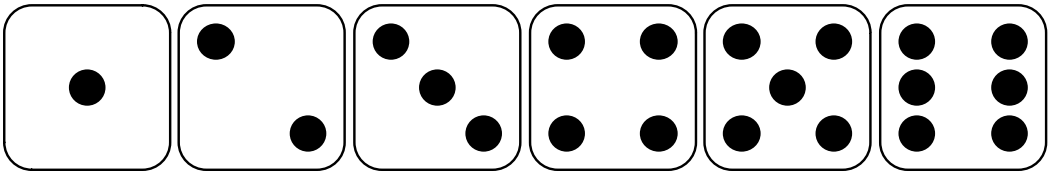
4 Mathematical Distributions

4.1 Basic Statistical Terms

An introduction to the basic procedures of statistics generally begins with the definition of the random experiment: An experiment with a non-predictable outcome is called a random experiment. All possible outcomes of a random experiment are elements of an "event space". The random variable is a depiction (or function) which allocates a real number to each element of this set.

These terms seem somewhat displeasing at first. They will, however, become quickly understandable on the basis of an example.

The toss of a (standard) die is a random experiment. The outcomes of such tosses are given through the symbols which are visible on the side of the die facing upwards as soon as the die has settled.



The set of these six symbols (toss outcomes, events) is the "event space". The random variable X ("dot count") allocates a real number to each element of this set. X can, then, assume the values 1, 2, ... , 6.

If we are dealing with a die game where the highest possible "dot count" is to be reached, and player A has already tossed a "4", then the toss outcomes of "5" and "6" are favorable events for player B (because he will win). From such a game situation, the classic definition of probability (according to Bernoulli and Laplace) can be derived:

$$P = \frac{\textit{Number of all favorable cases}}{\textit{Number of all possible cases}} = \frac{g}{m}$$

The "favorable" or "possible" cases naturally always refer to the "experiment" being considered at the moment. In the given example, the probability for player B to win is:

$$P = \frac{g}{m} = \frac{2}{6}.$$

If one applies these statistical terms to a production process, the production or processing of a raw material or unworked piece corresponds to a random experiment. All possible outcomes of the process (manufactured parts) are elements of the "event space". The random variable allocates a real number to each element of the "event space", for example, a dimension for the characteristic of diameter with the measurement unit being mm.

Contrary to the example of the die game, the "event space" can neither be described in advance (a-priori) on the basis of knowledge of the machine functions and process or general conditions before a machine capability study nor (a-posteriori) after a machine capability study on the basis of a finite number of measured characteristic values.

The elements of an “event space” are, of course, seen physically, not really random (any production planning would then be nonsensical) and the random variable of diameter can merely assume a rational value between zero and a finite, positive number given by the machine size or the maximum size of the unworked piece. Two different values (realizations) of the random variable must hereby distinguish themselves by at least the resolution of the measurement instrument or measurement procedure.

Strictly speaking, then, the random variable is not the diameter of the manufactured components, but rather the measurement value indicated by the measurement instrument at the moment of the reading.

Because of the non-definable “event space” and the non-equiprobable events due to the machine functions, the definition of probability according to Bernoulli and de Laplace (*P*=number of favorable cases/number of possible cases) can clearly not be applied. On the basis of the process experience with a “large” number of manufactured parts, an a-posteriori probability can be determined, for example in the form of the share of characteristic values which are allotted to the jth class (a half-open interval, then) of grouping K (given by way of the class limits). The mathematical probability is then equated in this case to a relative frequency (empirical probability). It is consequently a number between zero and one.

Corresponding to Table 2.3 and Fig. 2.2, one can make the following statements, for example, for the process considered there (in consideration of the explanation in Section 4.2):

- 92.8 % of the sample values (deviations of target value C) lie between $-3.217 \cdot \frac{1}{100}\,\text{mm}$ and $+2.617 \cdot \frac{1}{100}\,\text{mm}$.
- The probability that the random quantity X (deviation from target value C) will assume a value within the interval $\left[-0.03217\text{mm}; +0.02617\text{mm}\right]$ is 92.8 %.

Although both of these statements appear quite similar, there is a considerable difference between them. While the first statement refers to the concrete, present measurement values, a transition to a fictitious random quantity takes place in the second statement whose realizations are the observed measurement values x_i.

On the basis of Table 2.3 and Fig. 4.1, it becomes comprehensible how one reaches the first statement. In the bottom-most line of Table 2.3, the accompanying cumulative frequency is given for each class. These numbers are produced when the relative frequencies from the first class to (and including) this class are summed up.

The statement of 59.5 % cumulative relative frequency for the 4th class signifies that 59.5% of the values of the data set lie below the upper class limit of the 4th class (2.4 % + 9.5 % + 19.0 % + 28.6 % = 59.5 %). The same information can be taken from Figure 4.1.

One reaches the first of the above statements by subtracting the cumulative relative frequency of the first class from the cumulative relative frequency of the sixth class (95.2 % - 2.4 % = 92.8 %).

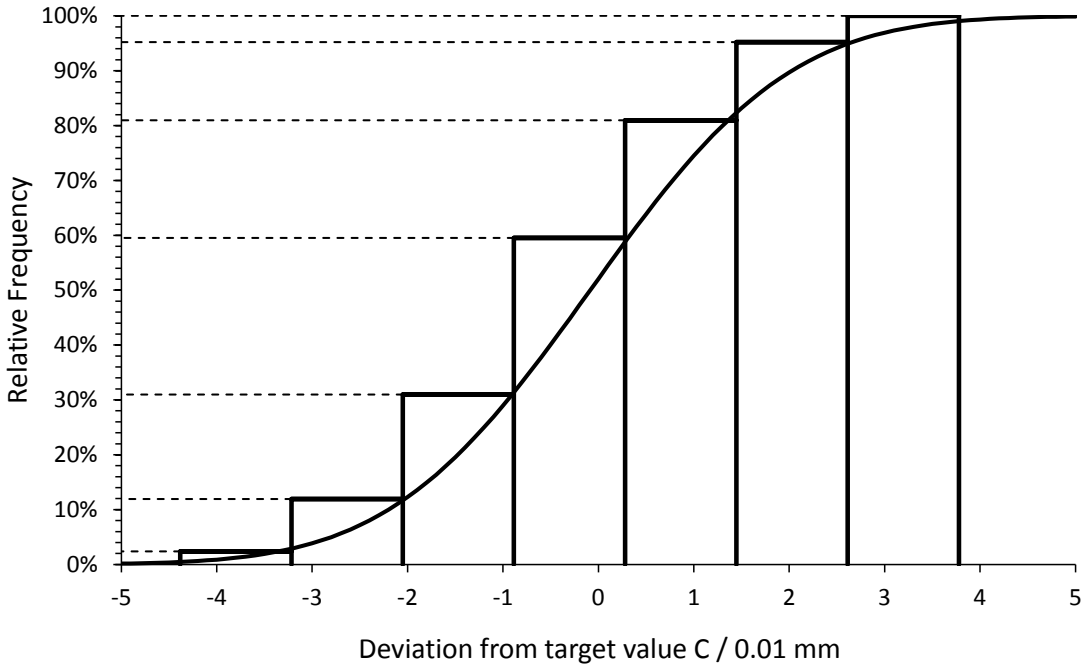


Fig. 4.1: Representation of the cumulative relative frequency (solid line: normal distribution)

If one interprets the measurement values from Table 2.2 as realizations of a random value X, then the relative frequency of a class gives the probability with which a value of X will be allotted to this class. The relative frequency can be understood, then, as a probability function, and a histogram can be understood as a graphic representation of this function.

A function $F(x)$, which gives the probability P with which a random variable X will assume values for each real number which are smaller than x is called the distribution function of the random variable X.

The distribution function is therefore defined by the equation $F(x) = P(X \leq x)$.

Correspondingly, the cumulative relative frequency is to be equated to a distribution function. Fig. 4.1 shows a graphic representation of this distribution function.

The following, for example, can be taken from it (and from Table 2.3):

$$F(2.617) = P(X \leq 2.617) = 95.2\% \quad \text{and} \quad F(-3.217) = P(X \leq -3.217) = 2.4\%.$$

From this follows directly the above second statement:

$$P(-3.217 \leq X \leq 2.617) = F(2.617) - F(-3.217) = 95.2\,\% - 2.4\,\% = 92.8\,\%.$$

X is the deviation (in $\frac{1}{100}$ mm) from the target value C (midpoint of the tolerance zone) in this example.

If all limitations to the measurement quantity X are neglected which are inevitable produced by the production conditions and the measurement procedure, and if X is considered exclusively as a real number (random variable), which can assume any values between $-\infty$ and $+\infty$, then one finally reaches a purely mathematically defined probability term.

The probability is produced by the integration of a probability density function over an interval. The term “density” is a reminder of the analogy between the probability calculation and the mechanics of rigid bodies (see e.g. [13]).

The solid curves in Fig. 2.2 and Fig. 4.1 represent the density function (Fig. 2.2) and the distribution function (Fig. 4.1) of the normal distribution (see Section 4.1).

The transition from “real” measurement values to a fictitious random quantity and its distribution opens the gate, so to say, to inductive statistics, whose terms and procedures are the subject of the following sections.

It is clear that one is placing himself in an isolated world of mathematical models. These can definitely be helpful to the practician in many cases in the evaluation and assessment of measurement series. On the other hand, however, it should not be forgotten that the information gained from these models are always associated with the models, and to transfer these to practical problems without thinking can lead to completely false conclusions.

4.2 Normal Distribution

If a normal distribution is being spoken of, one usually associates this term with the Gaussian curve. The Gaussian bell-shaped curve is a representation of the probability density function $f(x)$ of the normal distribution:

$$f(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot \left(\frac{x-\mu}{\sigma}\right)^2} .$$

The normal distribution gives the probability for each value x that the random variable X will assume a value between $-\infty$ and x. One obtains the distribution function $F(x)$ of the normal distribution by integrating the density function given above:

$$F(x) = \frac{1}{\sigma \cdot \sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-\frac{1}{2} \cdot \left(\frac{v-\mu}{\sigma}\right)^2} dv .$$

$F(x)$ corresponds to the area below the Gaussian curve up to the value x.

The special significance of the normal distribution in statistics is founded via the central limit theorem. Formulated somewhat freely, it states that through the random interaction (addition) of many independent random variables a random variable is produced which is approximately normally distributed.

If one formally considers the factors associated with a production process — man, machine, material, method, milieu (“the 5 Ms”) — as independent random variables, then the central limit theorem explains the fact that one will frequently find a distribution of the characteristic values of the manufactured parts during a process study which can at least be approximately described by the Gaussian normal distribution (apart from processes with systematic changes of the process average such as tendencies and batch discontinuities).

The normal distribution is clearly fixed by both parameters μ (average) and σ (standard deviation). μ determines the position of the distribution on the x-axis, and σ determines its width (compare with Fig. 4.2).

Both of these theoretical parameters are, naturally, not known in a concrete situation in which a data set is to be described by a normal distribution, but must rather be estimated on the basis of the data on hand (samples).

The best possible estimates $\hat{\mu}$ and $\hat{\sigma}$ for μ and σ are given by the arithmetic mean $\bar{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$ and the standard deviation $s = \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(x_i - \bar{x}\right)^2}$ (n = sample size).

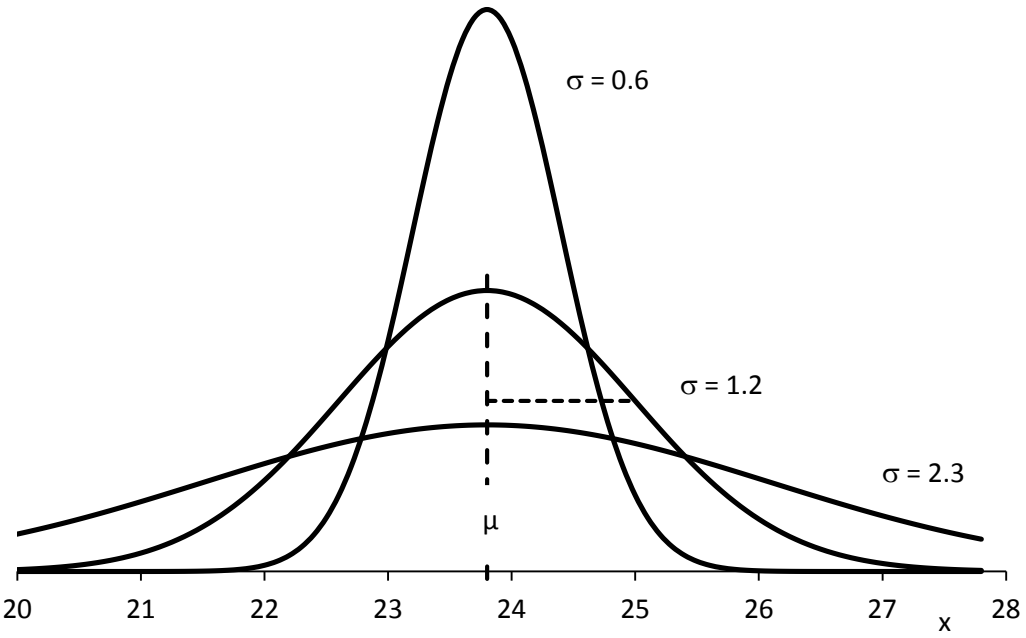


Fig. 4.2: Density functions of the normal distribution with $\mu = 23.8$ and standard deviations $\sigma = 0.6$, $\sigma = 1.2$ and $\sigma = 2.3$

A normally distributed random quantity X with the average $\hat{\mu}$ and standard deviation $\hat{\sigma}$ is converted to an equally normally distributed random quantity U through the transformation $U = \frac{X - \hat{\mu}}{\hat{\sigma}}$. The average of U is zero, and its standard deviation is one. The special normal distribution N(0,1) is called standard normal distribution..

The distribution function $\Phi(u)$ gives the probability with which the random quantity U will assume a value between $-\infty$ and u. $\Phi(u)$ corresponds to the area under the Gaussian curve all the way to value u. The total area under the bell-shaped curve has the value of one. The values for $\Phi(u)$ can be taken from the table.

With the assistance of the tabulated standard normal distribution, proportions nonconforming of any normal distribution can, for example, be calculated in reference to pregiven specification limits.

EXAMPLE 4.1:

The limits $LSL = 0.63\,mm$ and $USL = 0.71\,mm$ are given for the characteristic of a valve seat.

From the measurement values of a sample of 125 valves, the average $\hat{\mu} = \bar{x} = 0.674\,mm$ and the standard deviation $\hat{\sigma} = s = 0.0166\,mm$ were produced.

On the basis of these values, the proportion of the population which go below the lower limiting value or exceed the upper limiting value are to be calculated under the prerequisite of a normal distribution.

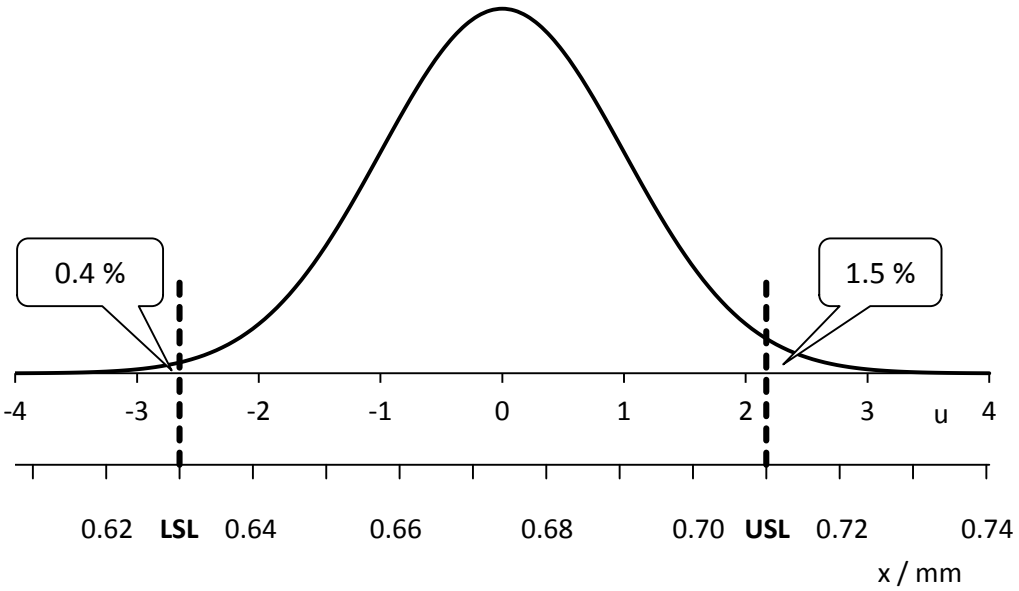


Fig. 4.3: Scale transformation for the calculation of proportions nonconforming with the assistance of the standard normal distribution

The following is produced in this example: $u = \frac{LSL - \bar{x}}{s} = \frac{0.63 - 0.674}{0.0166} = -2.65$.

One finds the following table value: $\Phi(-2.65) = 0.004$.

The proportion of all X which are smaller than $LSL = 0.63\,mm$ corresponds to the proportion of all U which go below the value $u = -2.65$. This is 0.4 % of the population.

The following is produced analogous to this: $u = \frac{USL - \bar{x}}{s} = \frac{0.71 - 0.674}{0.0166} = 2.17$.

Table value: $\Phi(2.17) = 0.015$

The proportion of all X which are greater than $USL = 0.71\,mm$ corresponds to the proportion of all U which exceed the value $u = 2.17$. This is 1.5 % of the population.

4.3 Confidence Interval of Proportions Nonconforming

The estimation of one-sided or two-sided proportions nonconforming with the assistance of the standard normal distribution contains several “error sources”. The quantities $\bar{x}$ and s used in the calculation of proportions nonconforming are estimations of the unknown characteristics μ and σ of the normal distribution. From sample to sample, different values for $\bar{x}$ and s can be produced, despite an unchanged population.

If one now calculates a proportion nonconforming as described in the example in 4.2 in reference to the upper specification limit USL on the basis of the dimensionless auxiliary quantity $u = \frac{USL - \bar{x}}{s}$, then a random variation range for u and thereby one for the proportion nonconforming $p = 1 - \Phi(u)$ is produced due to the variations of $\bar{x}$ and s.

One could tend to insert the limits of the confidence interval for μ and σ handled in the Sections 6.4 and 6.5 for u in the formula, to calculate the smallest possible and the greatest possible u, and to determine a confidence interval for p. This procedure would, however, not be correct, statistically speaking, since both estimations $\bar{x}$ and s are calculated from the same measurement values and are therefore not random quantities which are independent of one another.

The confidence interval for p is instead determined with the assistance of the non-central t-distribution [18]. It can be taken from the nomogram according to Fig. 4.4 dependent on the sample size n and the interval of the average $\bar{x}$ of one of the limiting values $a = LSL$ or $a = USL$ given as a multiple of the standard deviation s.

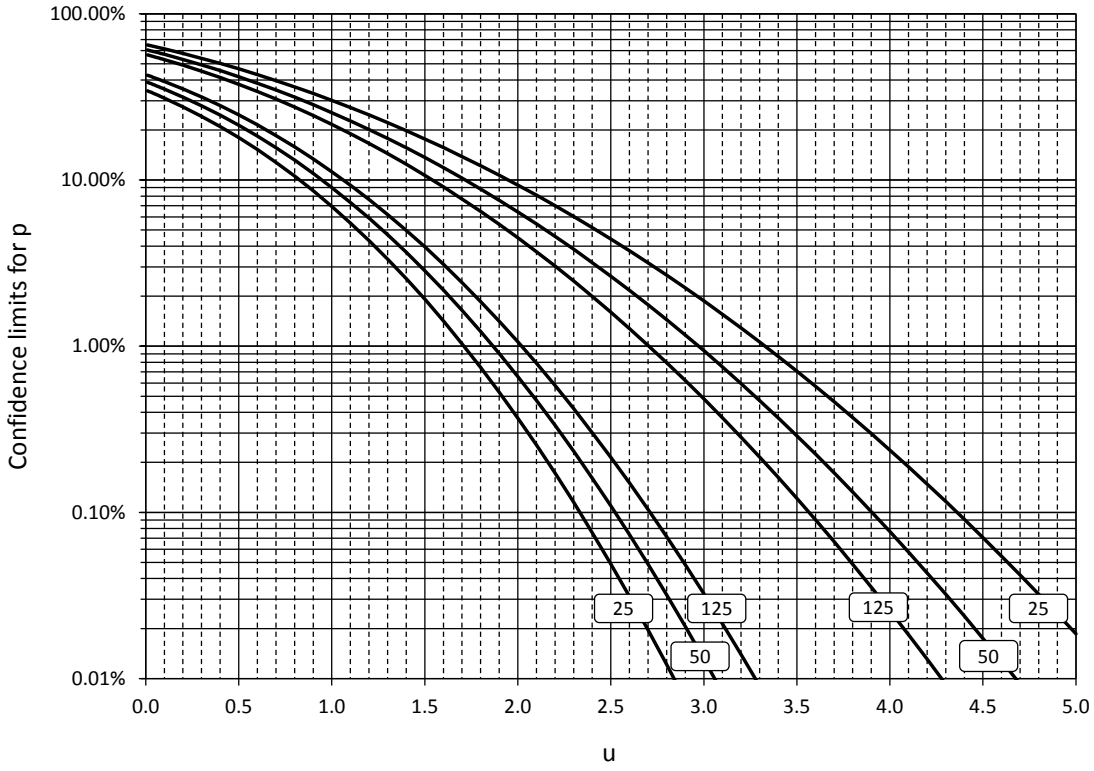


Fig. 4.4: Representation for the determination of (two-sided) 95 % confidence limits for the proportion nonconforming p in reference to a limit a. Sample size n as parameter.

In the example from Section 4.2, the interval of the average from the upper limit value is

$u = \frac{a - \bar{x}}{s} = \frac{0.71 - 0.674}{0.0166} = 2.17$. *The sample size is* $n = 125$.

According to Fig. 4.4, the interval [0.6%; 3.2%] is produced in an approximated manner as the confidence interval for p.

In correlation with the Statistical Process Control (SPC; compare with [5]), the process capability is assessed, as a rule, on the basis of the capability index C_{pk}.

C_{pk} is the smallest of the two values $\frac{\bar{x} - LSL}{3 \cdot s}$ and $\frac{USL - \bar{x}}{3 \cdot s}$.

Under the prerequisite of a normally distributed population, the C_{pk} value is usually directly associated with an accompanying proportion nonconforming p. Whether or not this association has a production-technical reference is an open question. The interval u considered above can also be given in this case in the form $u = 3 \cdot C_{pk}$. A confidence interval can also be determined for C_{pk} in a similar manner as p. This can be easily determined on the basis of Figure 4.5 or Table 4.1.

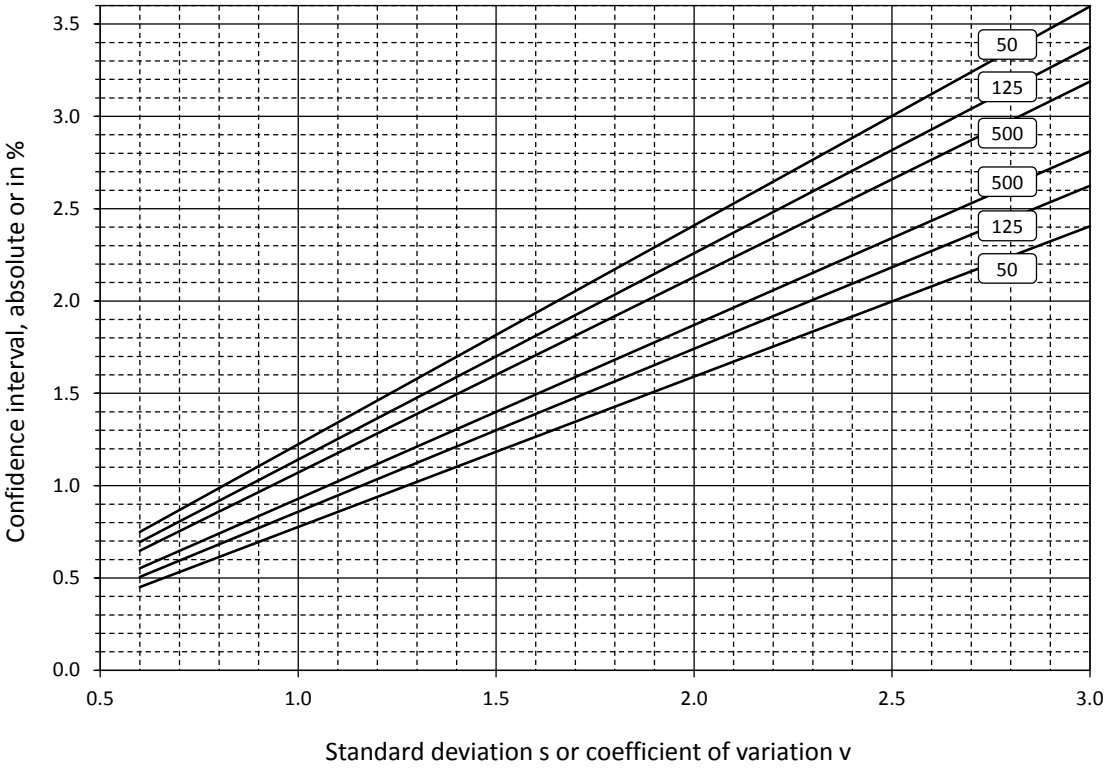


Fig. 4.5: Representation for the determination of (two-sided) 95 % confidence intervals for C_{pk}. Sample size n as parameter.

	$\hat{C}_{pk}=1.0$		$\hat{C}_{pk}=1.33$		$\hat{C}_{pk}=1.67$		$\hat{C}_{pk}=2.0$	
n	C_{pk}^{lower}	C_{pk}^{upper}	C_{pk}^{lower}	C_{pk}^{upper}	C_{pk}^{lower}	C_{pk}^{upper}	C_{pk}^{lower}	C_{pk}^{upper}
50	0.78	1.22	1.04	1.62	1.31	2.03	1.57	2.43
60	0.80	1.20	1.07	1.59	1.34	2.00	1.60	2.40
70	0.82	1.18	1.09	1.57	1.36	1.98	1.63	2.37
80	0.83	1.17	1.10	1.56	1.38	1.96	1.66	2.34
90	0.84	1.16	1.12	1.55	1.40	1.94	1.68	2.32
100	0.85	1.15	1.13	1.53	1.41	1.93	1.69	2.31
110	0.85	1.15	1.14	1.52	1.43	1.91	1.71	2.29
120	0.86	1.14	1.14	1.52	1.44	1.90	1.72	2.28
130	0.87	1.13	1.15	1.51	1.45	1.89	1.73	2.27
140	0.87	1.13	1.16	1.50	1.45	1.89	1.74	2.26
150	0.88	1.13	1.16	1.50	1.46	1.88	1.75	2.25
200	0.89	1.11	1.19	1.47	1.49	1.85	1.78	2.22
500	0.93	1.07	1.24	1.42	1.56	1.78	1.86	2.14

Table 4.1: Confidence limits for C_{pk}

The representations in Figures 4.4 and 4.5 and Table 4.1 were calculated with the assistance of the approximation formula given in [19].

Corresponding to [19] the limits (two-sided, $P_A=95\%$) of the confidence interval $[C_{pk}^{lower};C_{pk}^{upper}]$ for C_{pk} can be calculated as follows (see also [20]):

$$C_{pk}^{lower}=\hat{C}_{pk}\cdot\left(1-1{,}96\cdot\sqrt{\frac{1}{9\cdot n\cdot\hat{C}_{pk}^{2}}+\frac{1}{2\cdot n}}\right)$$

$$C_{pk}^{upper}=\hat{C}_{pk}\cdot\left(1+1{,}96\cdot\sqrt{\frac{1}{9\cdot n\cdot\hat{C}_{pk}^{2}}+\frac{1}{2\cdot n}}\right).$$

$\hat{C}_{pk}$ is therein the calculated estimation of the C_{pk} value by way of the characteristic values $\bar{x}$ and s of the sample, and n is the total sample size. Normal distribution of the population is a prerequisite.

4.4 Lognormal Distribution

A skewed distribution often occurs when a characteristic cannot go below (e.g. roughness, eccentricity, roundness) or exceed (e.g. flow rate, hardness) a limiting value.

For example, the characteristic of surface roughness can only assume values greater than zero. An asymmetric, zero-limited curve of the density function is produced which runs flat on the right side. If one applies a logarithm to such a distribution, he finds that the logarithmic values are approximately normally distributed. The application of a logarithm converts the number range between 0 and 1 into the range $-\infty$ to 0; the left portion of the distribution is, then, heavily stretched and the right portion is heavily compressed.

Definition:

A continuous random variable X is called lognormally distributed if $\ln(X)$ is normally distributed.

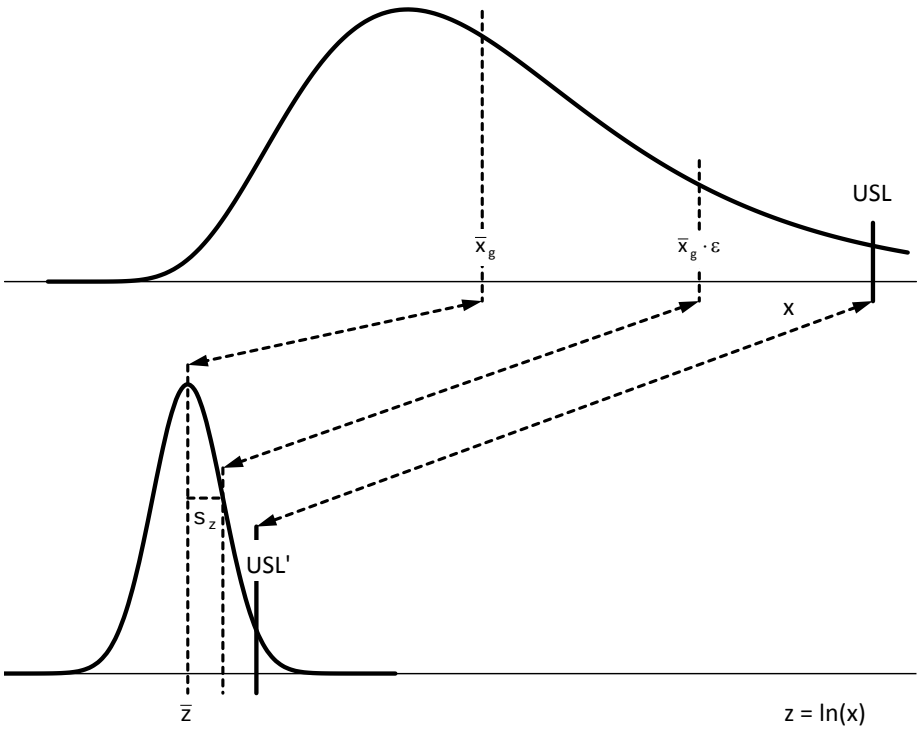


Fig. 4.6: Illustration of a lognormally distributed random variable

Fig. 4.6 should clarify that the function $z = \ln(x)$ converts lognormally distributed characteristic values x_i to normally distributed values z_i.

The expression $\ln(\bar{x}_g) = \bar{z} = \frac{1}{n} \cdot \sum_{i=1}^{n} z_i = \frac{1}{n} \cdot \sum_{i=1}^{n} \ln(x_i) = \ln\left(\sqrt[n]{\prod_{i=1}^{n} x_i} \right)$ defines the quantity $\bar{x}_g$, the geometric mean. With the normal distribution, respectively 50 % of all values lie below and above the average. This fact remains unchanged with the reconversion $x = e^z$. The quantity $\bar{x}_g$ therefore corresponds to the median of the lognormal distribution, and the following applies: $\bar{x}_g = \tilde{x} = e^{\bar{z}}$.

In Section 4.2, it is described how in cases of the normal distribution the proportion nonconforming with respect to an upper specification limit USL can be determined in a relatively simple manner. Here, only the average $\bar{x}$ and the standard deviation *s* are necessary in addition to USL.

With the lognormal distribution, the shape parameter ε (epsilon) plays a similar role to that of s with the normal distribution. If one assumes $s_z = \ln(\varepsilon)$ and $\varepsilon = e^{s_z}$ in a purely formal manner, the following relationship suggests a presumption as to how the variation limits of the distribution can be calculated:

$$\bar{z} + u \cdot s_z = \ln(\bar{x}_g) + u \cdot \ln(\varepsilon) = \ln(\bar{x}_g \cdot \varepsilon^u).$$

Between $\frac{\bar{x}_g}{\varepsilon}$ and $\bar{x}_g \cdot \varepsilon$ lie 68.3 % of all values of a lognormal distribution

Between $\frac{\bar{x}_g}{\varepsilon^2}$ and $\bar{x}_g \cdot \varepsilon^2$ lie 95.4 % of all values of a lognormal distribution

Between $\frac{\bar{x}_g}{\varepsilon^3}$ and $\bar{x}_g \cdot \varepsilon^3$ lie 99.7 % of all values of a lognormal distribution

1. The determination of specification limits LSL and USL in such a manner that respectively α% of all values lie outside of the interval $[\mathrm{LSL}, \mathrm{USL}]$.

   Calculation of the geometric mean $\bar{x}_g$ with the assistance of the average $\bar{z}$ of the logarithms of the values x_i: $\bar{z} = \frac{1}{n} \cdot \sum_{i=1}^{n} \ln(x_i)$, $\bar{x}_g = e^{\bar{z}}$.

   Calculation of the shape parameter ε with the assistance of the standard deviation of the logarithms of the x_i:

   $$s_z = \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(\ln(x_i) - \bar{z}\right)^2} = \sqrt{\frac{1}{n-1} \cdot \left[\sum_{i=1}^{n} \left(\ln(x_i)\right)^2 - n \cdot \bar{z}^2\right]} \qquad \varepsilon = e^{s_z}.$$

   Extraction of the quantity u with $\Phi(u) = 1 - \alpha$ and the quantity $-u$ with $\Phi(-u) = \alpha$ from the table of the standard normal distribution.

   $\mathrm{LSL} = \bar{x}_g \cdot \varepsilon^{-u} = \frac{\bar{x}_g}{\varepsilon^u}$ and $\mathrm{USL} = \bar{x}_g \cdot \varepsilon^u$

2. Determination of a proportion nonconforming with respect to a pre-given limiting value

   Calculation of the geometric mean $\bar{x}_g$ and the shape parameter ε with assistance of the logarithms of the characteristic values x_i as in item 1.

Upper specification limit given: $USL = \bar{x}_g \cdot \varepsilon^u \quad \Leftrightarrow \quad u = \dfrac{\ln\left(\dfrac{USL}{\bar{x}_g}\right)}{\ln(\varepsilon)}$

Lower specification limit given: $LSL = \bar{x}_g \cdot \varepsilon^{-u} = \dfrac{\bar{x}_g}{\varepsilon^u} \quad \Leftrightarrow \quad u = -\dfrac{\ln\left(\dfrac{LSL}{\bar{x}_g}\right)}{\ln(\varepsilon)}$

Extraction of the proportion nonconforming α with $\alpha = 1 - \Phi(u)$ from the table of the standard normal distribution.

Example for 2.:

In connection with a flanging process, torque measurements are conducted. With ten of the parts inspected, the measurement values x_i were produced and are given in the following table. The proportion of the parts of the population whose characteristic values go below the lower specification limit $LSL = 400\,N\,cm$ is to be determined.

Part No.	$x_i / N \cdot cm$	$z_i = \ln(x_i)$
1	*540*	*6.29*
2	*745*	*6.61*
3	*580*	*6.36*
4	*605*	*6.40*
5	*900*	*6.80*
6	*655*	*6.48*
7	*705*	*6.56*
8	*480*	*6.17*
9	*810*	*6.70*
10	*690*	*6.54*

The following is produced: $\bar{z} = 6.49$ *and* $s_z = 0.19$

and thereby: $\bar{x}_g = e^{\bar{z}} = e^{6.49} = 659$, *as well as* $\varepsilon = e^{s_z} = e^{0.19} = 1.21$.

$$u = -\frac{ln\left(\frac{LSL}{\bar{x}_g}\right)}{ln(\varepsilon)} = -\frac{ln\left(\frac{400}{659}\right)}{ln(1.21)} = 2.62 \qquad \alpha = 1 - \Phi(2.62) = 1 - 0.9956 = 0.0044$$

The lower specification limit is gone below by 0.44% of the parts of the population.

4.5 Lognormal Probability Plot

The lognormal probability plot contains the same subdivision of the y-axis as the conventional probability plot (see [1]); however, it also contains a logarithmic subdivision of the x-axis.

In using this plot type, a data set can be inspected in a simple manner with regards to a lognormal distribution of the values. Should the assumption of a lognormal distribution be justified, the parameters $\bar{x}_g$ and ε as well as proportions nonconforming regarding given limiting values can be additionally graphically determined.

As long as the number of the given values is large enough, one can create a histogram of these values and determines, then, the relative frequency of values within the classes of a grouping. If one records the relative cumulative frequencies over the right class limit in the probability plot and a sequence of points is thereby produced which lie in an approximated manner on a line, one can conclude from this that the values of the data set are approximately lognormally distributed.

If the number n of the present values x_i are not adequate for the creation of a histogram, one can allocate cumulative frequencies to the individual values x_i by way of the following procedure so that a recording in the lognormal probability plot is also possible in this case.

1. The values $x_1, x_2, \ldots, x_n$ are arranged in order of magnitude:

   $x_{(1)} \leq x_{(2)} \leq \ldots \leq x_{(n)}$.

   The smallest value $x_{(1)}$ has rank 1, and the largest value $x_{(n)}$ has rank n.

2. Every $x_{(i)}$ $(i = 1, 2, \ldots, n)$ is allocated a relative cumulative frequency $H_i(n)$:

   $x_{(1)}$, $x_{(2)}$, ... , $x_{(n)}$

   $H_1(n)$, $H_2(n)$, ... , $H_n(n)$.

   The cumulative relative frequency $H_i(n)$ for rank number i can be calculated with one of the approximation formulas

   $$H_i(n) = \frac{i - 0.5}{n} \qquad \text{and} \qquad H_i(n) = \frac{i - 0.3}{n + 0.4}$$

   The deviation from exact table values is insignificant here.

3. Representation of the points $(x_{(i)}, H_i(n))$ in a probability plot.

We will assume the values from the above example (torque measurement) for the demonstration of this procedure and arrange them in order of magnitude.

Priority i	x_i / N · cm	$H_i(10)$
1	*480*	*6.2 %*
2	*540*	*15.9 %*
3	*580*	*25.5 %*
4	*605*	*35.2 %*
5	*655*	*45.2 %*
6	*690*	*54.8 %*
7	*705*	*64.8 %*
8	*745*	*74.5 %*
9	*810*	*84.1 %*
10	*900*	*93.8 %*

The points (x_i, H_i) *are recorded in the lognormal probability plot.*

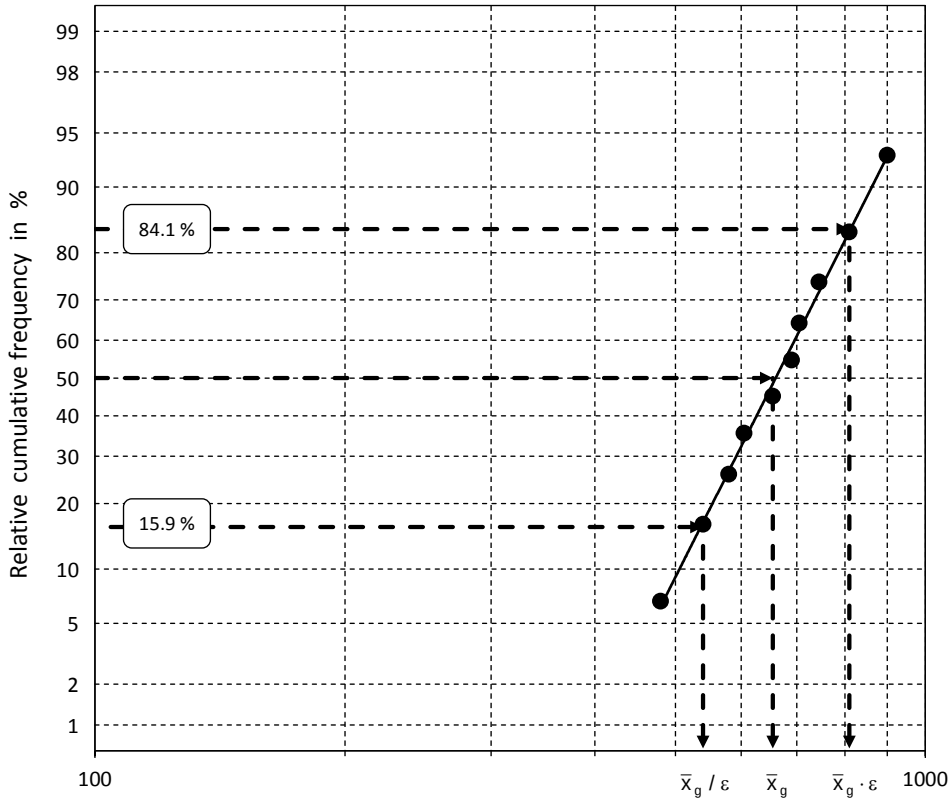


Fig. 4.7: Recording of the measurement values in the lognormal probability plot.
x-axis: torque in N·cm

If one draws a compensating line through the points, the estimated values $\frac{\bar{x}_g}{\varepsilon}$, $\bar{x}_g$ *and* $\bar{x}_g \cdot \varepsilon$ *can be read as abscissa values of the ordinate values 5.9 %, 50 % and 84.1 %.*

In a similar manner, estimated values can be read directly for proportions nonconforming or proportions going below with respect to a given limit, or a limiting value can be set on the basis of a given proportion nonconforming. In all cases, however, it must be clear that such estimates are always full of uncertainties which become greater with fewer measurement values present. “Extrapolations” through extensions of the compensating lines beyond the ranges covered by the measurement points should particularly be avoided.

Special Cases

If it is not possible to approximate the points in the lognormal probability plot by way of a straight line, but it is possible by way of a curved line to the x-axis, it is possible that the quantity X is not lognormally distributed, but rather X - a. In this case, the characteristic X possesses a natural limit a > 0, which cannot be gone below (compare with [10] p. 98).

If the points can be approximated well by a curved line moving out from the x-axis in the log probability plot, a - X is possibly lognormally distributed. In this case, the characteristic X possesses a natural limit a > 0, which cannot be exceeded. Examples of such characteristics with a natural upper limit are the hardness of a metal surface and the tensile strength of a welded joint.

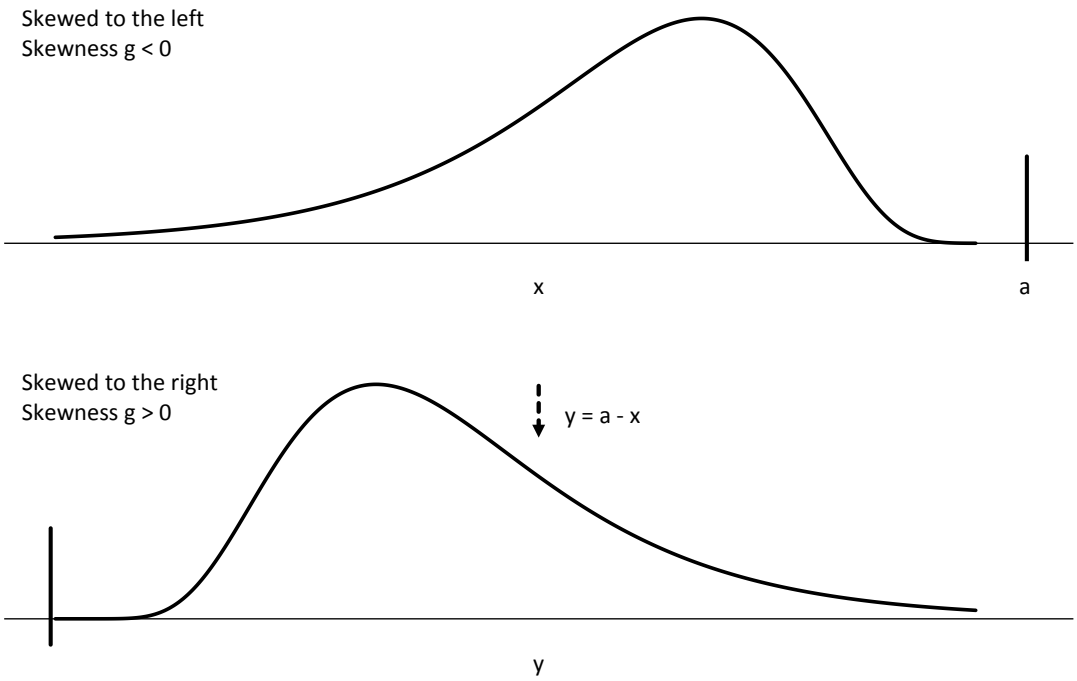


Fig. 4.8: Conversion of a lognormal distribution which is skewed to the left to one which is skewed to the right

In both cases the question is how quantity a is to be selected. As long as a is not known due to the technical facts, an estimated value for a must be determined. Here, the abscissa values (on the x-axis) x, x' and x'' for the relative cumulative frequency $H = 50\%$, $H' = 15.9\%$ and $H'' = 84.1\%$ are read at the curve in the lognormal probability plot.

The estimated value $\hat{a} = \left| \frac{x^2 - x' \cdot x''}{x' + x'' - 2 \cdot x} \right|$ is finally calculated.

Through the conversion $Y = a - X$ and $Y = X - a$, one obtains, then, a lognormally distributed quantity Y, to which the formulas of Section 4.4 can be applied.

Should limiting values for Y be calculated in this manner, they must afterwards be reconverted to limiting values for X.

COMMENT:

In the formula for the density function of the lognormal distribution

$$f(x) = \frac{1}{x \cdot \sigma \cdot \sqrt{2\pi}} \cdot e^{-\frac{1}{2} \cdot \left(\frac{ln(x) - \mu}{\sigma} \right)^2},$$

the natural (base e) logarithm $ln(x)$ occurs. Correspondingly, the given relations $\bar{x}_g = e^{\bar{z}}$ and $\varepsilon = e^{s_z}$ are produced in Section 4.3.

If one uses the common (base 10) logarithm $log(x)$ instead of the natural one as well as its inverse function 10^x, one finds the expressions $\bar{x}_g = 10^{\bar{z}}$ and $\varepsilon = 10^{s_z}$ (compare with e.g. [17]).

The x-axis of the log probability plot is subdivided in correspondence with the common logarithm. This fact is, however, insignificant with respect to the comparability with numerical calculation in accordance with the formulas from Section 4.4, since in the calculation of $\bar{x}_g$ and ε, a reconversion to the original coordinate system always occurs, and calculations of proportions nonconforming or limiting values are only conducted with assistance of both of these quantities.

4.6 Sample Characteristics

Sample outcomes (individual values) can be combined and represented by illustration in

- dot frequency diagrams
- gar graphs
- histograms and
- cumulative curves.

The information on the temporal order of the individual values, though, is lost. Statistical characteristics can be calculated from the individual values which characterize the distribution of the sample values.

These (empirical) characteristics are estimated values for the corresponding characteristics of the (theoretical) distribution of the (parent) population from which the sample was taken (compare with Table 4.2).

Characteristic		Population	Sample
of location:	Median	50 % value	$\tilde{x}$
	Arithmetic mean	μ	$\bar{x}$
	Geometric mean	μ_g	$\bar{x}_g$
of variability:	Variance	σ^2	s^2
	Standard deviation	σ	s
	Shape parameter	ε	ε
	Range	-	R
of shape:	Skewness	τ	g
	Excess	ε	e
	Kurtosis		g_w

Table 4.2: Usable designations of the characteristics for the population and the sample

4.6.1 Characteristics of Location

The following are usable as dimensions of the average location:

Characteristic	Calculation
Median n odd	$\tilde{x} = x_{\left(\frac{n+1}{2}\right)}$
n even	$\tilde{x} = \frac{x_{\left(\frac{n}{2}\right)} + x_{\left(\frac{n}{2}+1\right)}}{2}$
Arithmetic mean	$\bar{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$
Geometric mean	$\bar{x}_g = \sqrt[n]{\prod_{i=1}^{n} x_i} = \sqrt[n]{x_1 \cdot x_2 \cdot \ldots \cdot x_n}$

Table 4.3: Characteristics of location

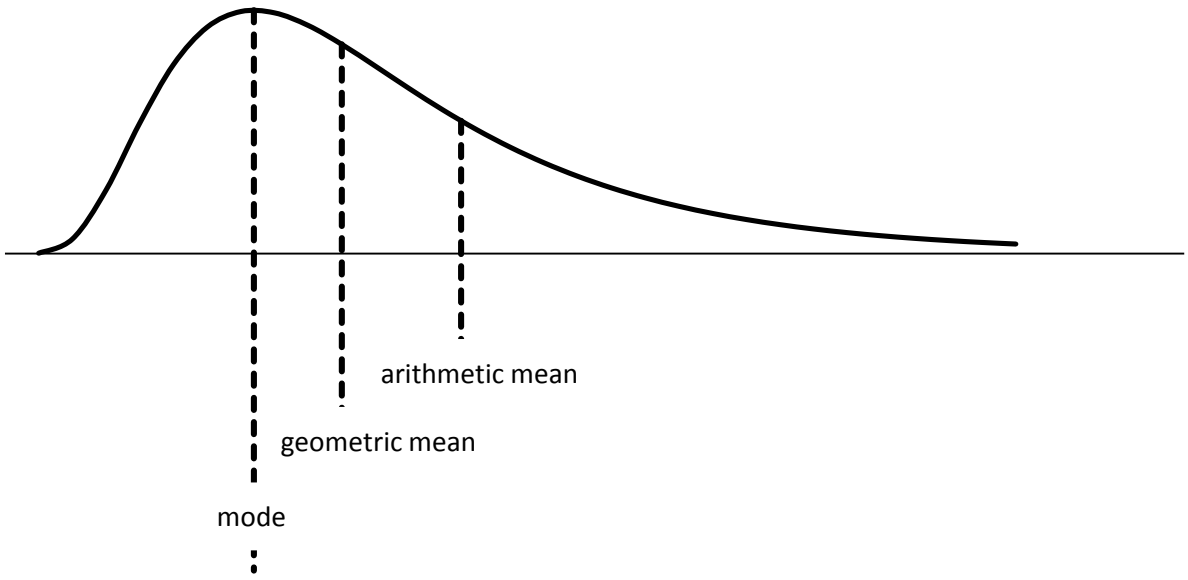


Fig. 4.9: Characteristics of location with a lognormal distribution

The median is the value within a sequence of individual values arranged in order of magnitude which halves the sequence. Due to its definition, the same number (50 %) of values of a data set always lie below and above the median (compare with [3]).

The most frequent value (the density average) of a (continuous, unimodal) distribution is the position at which the density function assumes its maximum. If we are dealing with a discrete distribution, the most frequent value corresponds to that with the greatest probability.

4.6.2 Characteristics of Variability

The variability is a measure for the deviations of characteristic values with respect to the average.

Characteristic	Calculation
Variance	$s^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^2$
Standard deviation	$s = \sqrt{s^2}$
Range	$R = x_{max} - x_{min}$
Coefficient of variation	$v = \frac{s}{\bar{x}} \cdot 100\,\%$
Shape parameter	$\varepsilon = e^s$ $s = \sqrt{\frac{1}{n-1} \cdot \sum_{i=1}^{n} \left(\ln(x_i) - \ln(\bar{x}_g)\right)^2}$

Table 4.4: Characteristics of variability

The standard deviation has a significance as the characteristic of variability similar to that of the arithmetic mean as the characteristic of location. It has the same unit of measure as the characteristic values and is calculated from the variance.

Since the range is calculated from only two values, its information content is fundamentally less than that of the standard deviation.

With the coefficient of variation, the extent of the variability of the individual values refers to the arithmetic mean. The variation coefficient is used, for example, in association with the giving of harmful substance concentrations (HC, CO and NO_x) in exhaust from combustion engines. The averages of these concentrations distinguish themselves very greatly from one another. The assessment of the effect of improvement measures on the variation of the individual values of the three types of harmful substance is made easier through the giving of the variation coefficients, since these are directly comparable.

The shape parameter is used as the characteristic of variability with the lognormal distribution. It is calculated from the standard deviation of the logarithms of the characteristic values (compare with Section 4.4).

4.6.3 Characteristics of Shape

Table 4.5 gives an overview of characteristics of “shape” of the distribution accompanying the measurement values. Strictly speaking, one thinks of the representation of the probability density function of the distribution when the term “shape” is used. The common usage is usually very inexact with respect to this.

Characteristic	Calculation
Skewness	$g = \frac{\frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^3}{s^3}$
Kurtosis	$g_W = \frac{\frac{1}{n} \cdot \sum_{i=1}^{n} (x_i - \bar{x})^4}{s^4}$
Excess	$e = g_W - 3$

Table 4.5: Characteristics of shape

The skewness is a measure of the asymmetry of a distribution. If the steep curve incline of a distribution lies to the left of the average, then the value is positive (the distribution is skewed to the right); in the other case, it is negative (the distribution is skewed to the left, compare with Fig. 4.8). The greater the amount of the skewness, the less symmetric the accompanying distribution will be.

An examined population approximately underlies a normal distribution if the limiting values for the skewness g (dependent on sample size n) given in Table 4.6 are not exceeded.

Sample size n	Limits for the skewness g, $P_A = 95\%$
10	0.95
15	0.85
20	0.77
25	0.71
30	0.66
40	0.59
50	0.53
80	0.43
100	0.39
200	0.28
500	0.18
1000	0.13

Table 4.6: Limiting values for the skewness g for sample size n

Since the interval $x_i - \bar{x}$ occurs to the third power in the expression for the skewness, an individual outlier can lead to a very great skewness, even though all other values are approximately normally distributed.

The kurtosis (flat- or peaked-'topness') and the excess are measures for how greatly flattened or pointed the representation of the density function of a distribution appears in comparison to the density function of the normal distribution (see Fig. 4.10).

The excess is equal to the kurtosis minus three.

The kurtosis of the normal distribution has the number value of three, and the excess of this distribution is correspondingly zero.

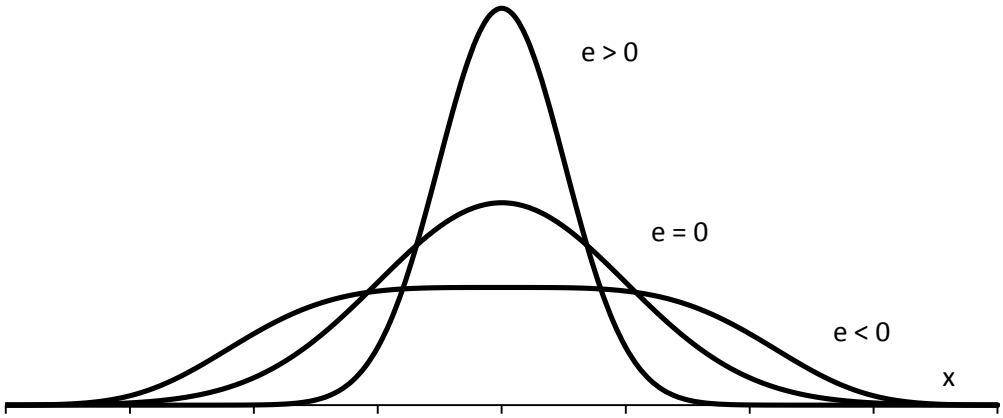
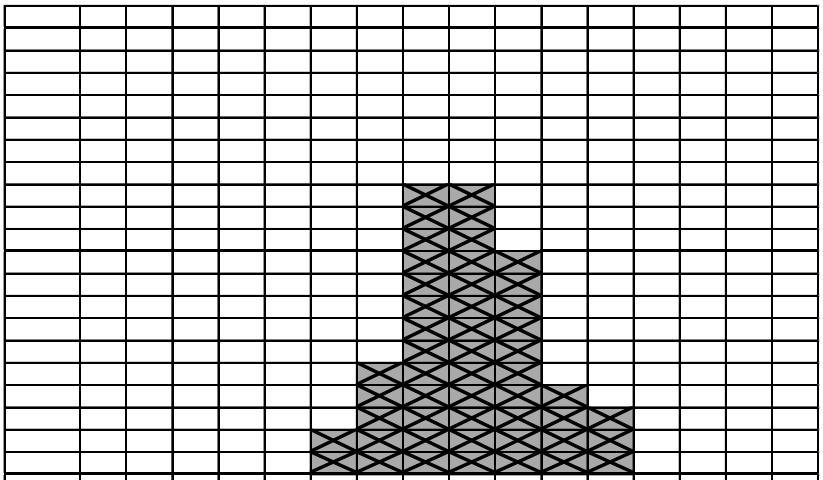


Fig. 4.10: Representation of the density functions of three distributions. $e = 0$ with the normal distribution.

4.6.4 Examples of Sample Characteristics

EXAMPLE 4.2:



n_j						2	5	13	13	10	4	3				
G_j						2	7	20	33	43	47	50				
H_j / %						4	14	40	66	86	94	100				

$m_1 = 7.3 = \bar{x}$

$m_2 = 1.36$

$m_3 = 0.53$

$m_4 = 9.14$

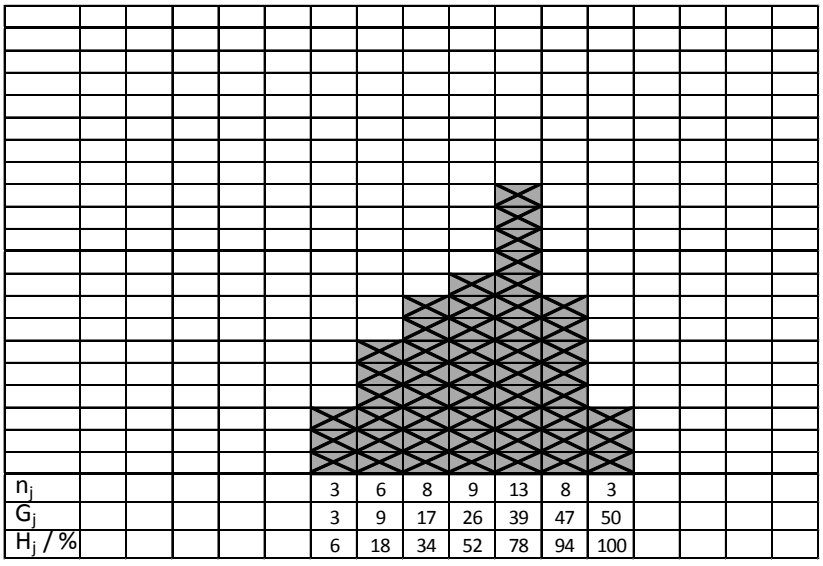
$g = 0.21$ *(skewness)*

$g_W = 2.6$ *(kurtosis)*

$e = -0.4$ *(excess)*

Fig. 4.11

EXAMPLE 4.3:



n_j						3	6	8	9	13	8	3				
G_j						3	9	17	26	39	47	50				
H_j / %						6	18	34	52	78	94	100				

$m_1 = 9.2 = \bar{x}$

$m_2 = 0.0142$

$m_3 = -2.42 \cdot 10^{-4}$

$m_4 = -4.0 \cdot 10^{-4}$

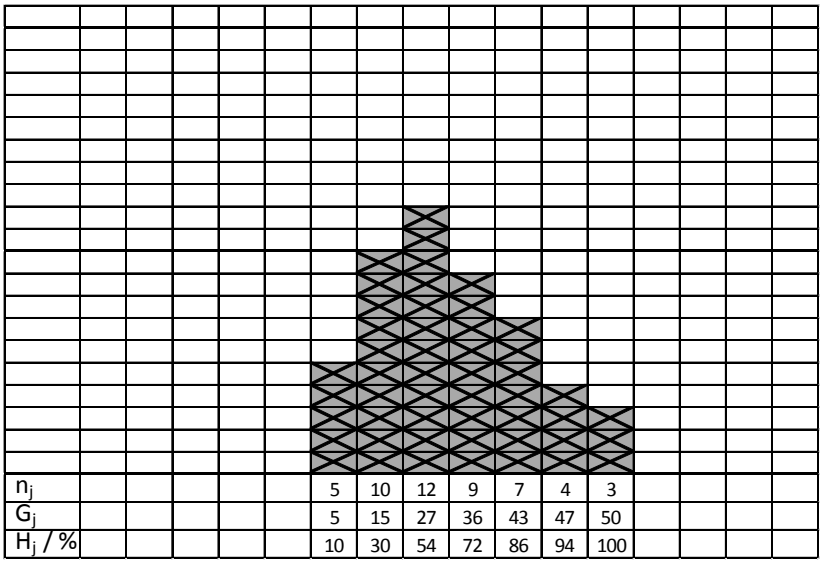
$g = -0.14$ *(skewness)*

$g_W = 2.0$ *(kurtosis)*

$e = -1.0$ *(excess)*

Fig. 4.12

EXAMPLE 4.4:



n_j						5	10	12	9	7	4	3				
G_j						5	15	27	36	43	47	50				
H_j / %						10	30	54	72	86	94	100				

$m_1 = 8.2 = \bar{x}$

$m_2 = 17.0$

$m_3 = 12.67$

$m_4 = 557$

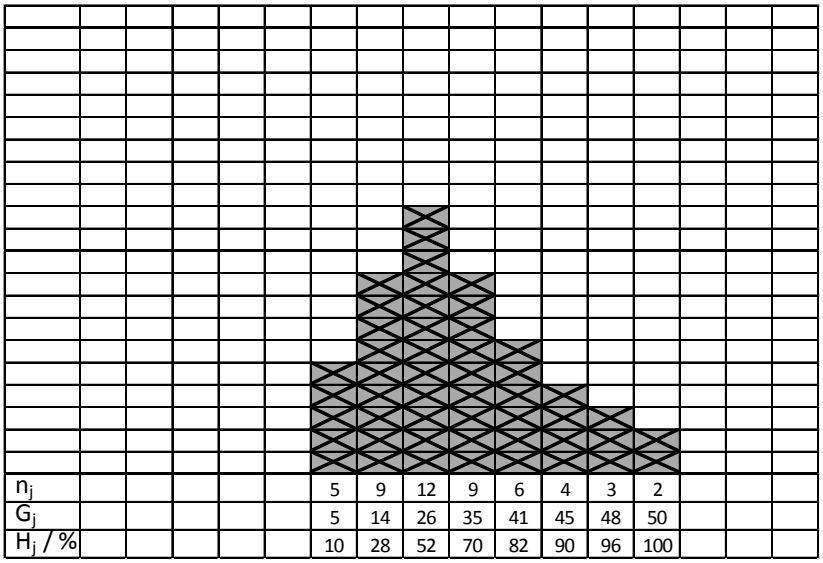
$g = 0.18$ *(skewness)*

$g_W = 1.9$ *(kurtosis)*

$e = -1.1$ *(excess)*

Fig. 4.13

EXAMPLE 4.5:



n_j						5	9	12	9	6	4	3	2			
G_j						5	14	26	35	41	45	48	50			
H_j / %						10	28	52	70	82	90	96	100			

$m_1 = 6.7 = \bar{x}$

$m_2 = 3.4$

$m_3 = 3.44$

$m_4 = 30$

$g = 0.55$ *(skewness)*

$g_W = 2.6$ *(kurtosis)*

$e = -0.4$ *(excess)*

Fig. 4.14

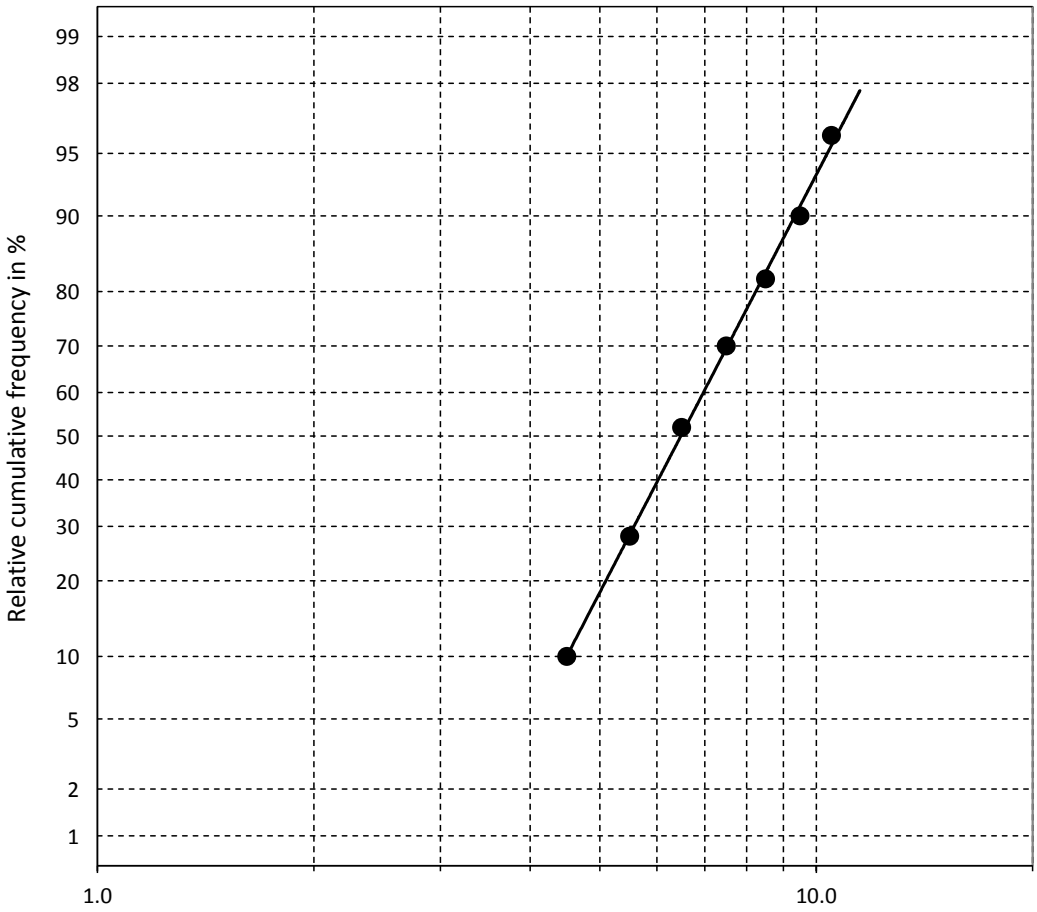
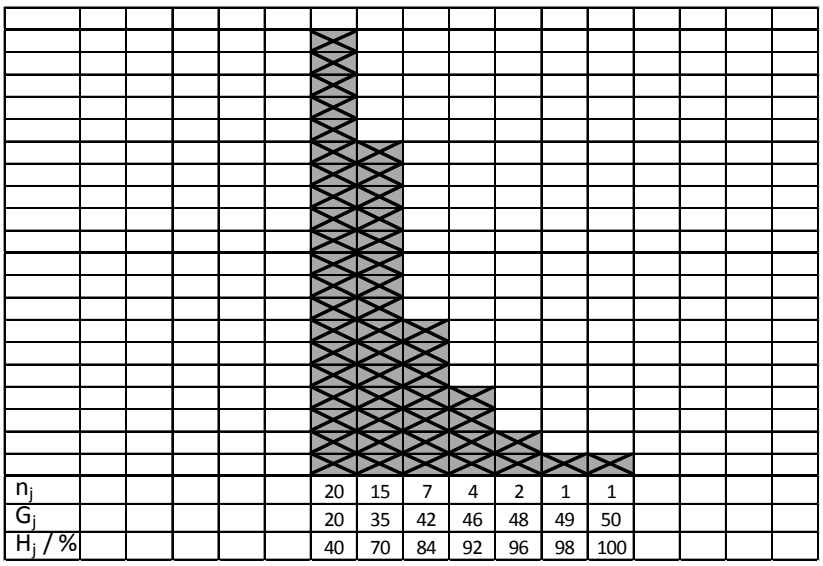


Fig. 4.15: Recording of the points from Example 4.5 in a lognormal probability plot.

EXAMPLE 4.6:



$m_1 = 6.7 = \bar{x}$

$m_2 = 3.4$

$m_3 = 3.44$

$m_4 = 30$

$g = 0.55$ *(skewness)*

$g_W = 2.6$ *(kurtosis)*

$e = -0.4$ *(excess)*

Fig. 4.16: Histogram for a one-sided limited characteristic

5 Direct and Indirect Inference

As shown in Section 4, statistics deal, on one hand, with distributions (statistical models) which can describe real or fictitious populations, and on the other, with characteristics of real samples.

In correspondence with both possible starting points (prerequisites), the following fundamentally different types of conclusions are produced:

1. Direct (deductive) inference on the sample from the population

   Prerequisite: The distribution of the population and its characteristics (parameters) are known (or are assumed as known).

   Typical applications can be found within the scope of probability calculation in association with the games theory, but also in probability statements in industrial practice.

   EXAMPLES:

   - *Chances of winning at roulette*
   - *Sampling plans for the examination of discrete characteristics (box model, compare with [2])*
   - *Calculation of proportions nonconforming in the calculation of tolerances*

2. Indirect (inductive) inference on the population from the sample

   Prerequisite: Individual outcomes of a sample are present which were taken from the population according to the principle of chance and can thereby be considered as representative for the population. The type of characteristic distribution (distribution function) of the population is assumed as known.

   EXAMPLES:

   - *Estimation of distribution parameters (point estimation) or their confidence intervals (interval estimation)*
   - *Comparison of parameters (statistical tests)*

The most significant methods and tests used here will be explained in the following sections. Only the mean μ or $\bar{x}$ and the standard deviation σ or s will be used as characteristics in these procedures.

Distribution of Sample Averages

The calculation of the distribution of averages provides a good starting point for the problem of the direct and indirect inference.

EXAMPLE 5.1: Length measurement

The following list of 50 original values is given:

Length /mm									
8.0	*7.4*	*6.5*	*7.0*	*6.0*	*7.5*	*8.3*	*8.0*	*5.8*	*9.0*
7.0	*7.8*	*7.5*	*7.5*	*8.0*	*8.4*	*7.3*	*7.5*	*7.8*	*8.0*
7.4	*7.5*	*7.6*	*7.1*	*7.0*	*6.8*	*7.0*	*7.0*	*6.3*	*7.1*
8.0	*7.7*	*7.3*	*7.4*	*8.0*	*8.3*	*7.5*	*6.5*	*7.5*	*7.0*
7.0	*6.9*	*8.0*	*8.6*	*6.9*	*8.0*	*7.9*	*7.8*	*7.9*	*7.4*

If we calculate the average from the 2, 3, 4, ... , n values with $\bar{x} = \frac{1}{n} \cdot \sum_{i=1}^{n} x_i$ *, starting with the first two values in the first row (8.0 and 7.4), we will obtain the following averages as functions of* n*:*

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14
$\bar{x}$	8.00	7.70	7.30	7.23	6.98	7.07	7.24	7.34	7.17	7.35	7.32	7.36	7.37	7.38

n	15	16	17	18	19	20	21	22	23	24	25	26	27	28
$\bar{x}$	7.42	7.48	7.47	7.47	7.49	7.52	7.51	7.51	7.51	7.50	7.48	7.45	7.43	7.42

n	29	30	31	32	33	34	35	36	37	38	39	40	41	42
$\bar{x}$	7.38	7.37	7.39	7.40	7.40	7.40	7.41	7.44	7.44	7.42	7.42	7.41	7.40	7.39

n	43	44	45	46	47	48	49	50
$\bar{x}$	7.40	7.43	7.42	7.43	7.44	7.45	7.46	7.45

In Fig. 5.1, these averages are marked with a dot.

If one starts with the last value (7.4) on the list in the calculation of averages, then he obtains the values marked with an "x".

The average of all 50 values on the list is 7.45 mm. In the example, the average calculated from 20 values is closer than that from 10 values, and this is, in turn, more accurate than the average calculated from 5 values. This applies for both the "dot curve" and the "x curve" (see Fig. 5.1).

The more measurement values that are brought into play in the calculation of the average, the more accurate the estimated value of the true calculated average is. This is a statistical regularity. The accuracy of the estimated values can also be given.

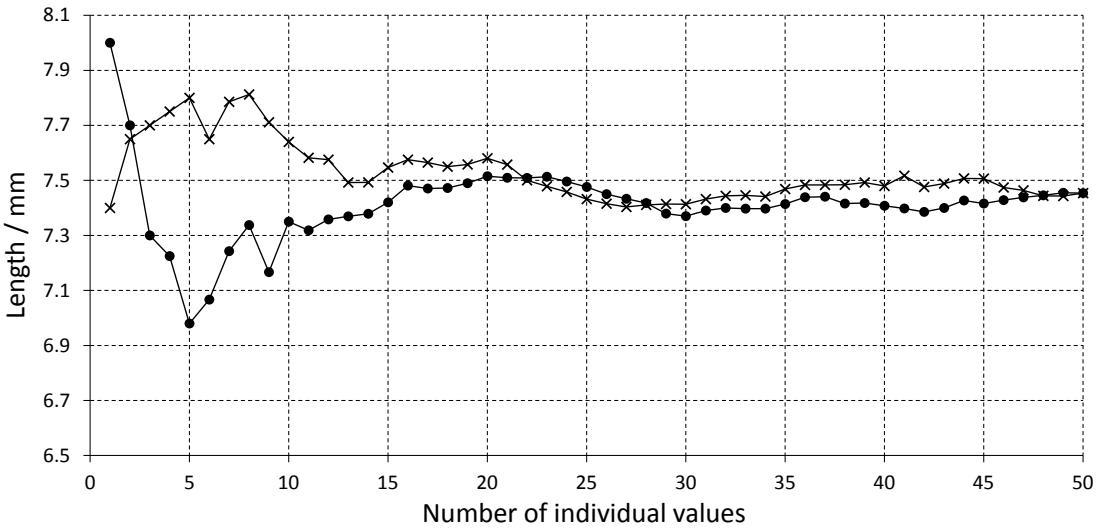


Fig. 5.1: Estimated values for the average

As suggested in Fig. 5.1, the estimated values vary around the average of the population. The extent of the variation is a measure of the accuracy of the estimated values with various n.

A characteristic for the variability of the averages is the standard deviation $\sigma_{\bar{x}}$. The standard deviation $\sigma_{\bar{x}}$ is calculated from the standard deviation σ of the individual values of the population according to the formula:

$$\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}.$$

n is the number of values brought into play in the calculation of the average $\bar{x}$.

If the n individual values originate from any distribution other than the normal distribution, a normal distribution (from roughly n = 4) is nevertheless approximately produced as the distribution of the averages (from the respective n values). The greater the sample size n, the better the approximation to the normal distribution (central limit theorem of statistics).

Correspondingly, 68.3 % of all averages $\bar{x}$ lie in the range between $\mu - \sigma_{\bar{x}}$ and $\mu + \sigma_{\bar{x}}$. This statement corresponds to a direct inference on the sample from the population.

With respect to the average μ and the standard deviation σ of the population, the statement can be made that the average of a sample of n individual values lies between $\mu - \frac{\sigma}{\sqrt{n}}$ and $\mu + \frac{\sigma}{\sqrt{n}}$ with a probability of 68.3 %.

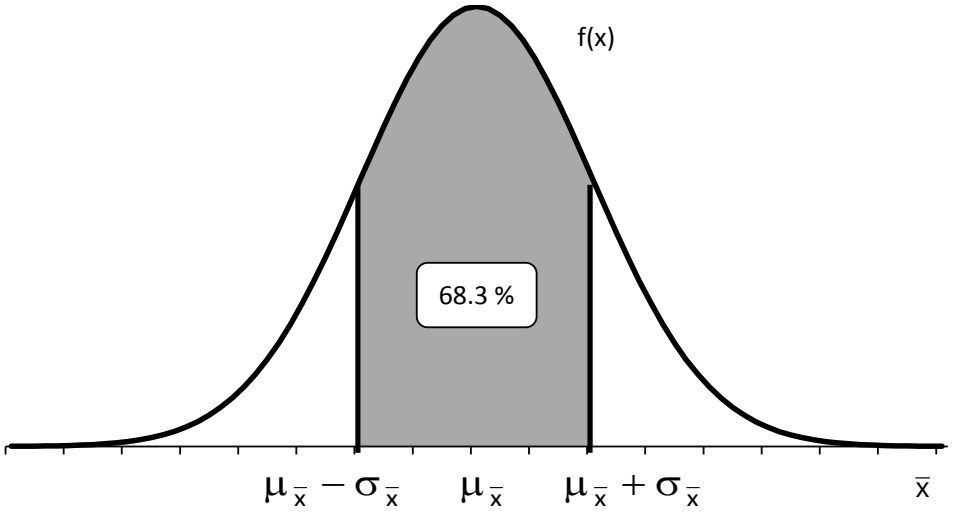


Fig. 5.2: Probability density function of the averages $\bar{x}$ calculated from n individual values

With the assistance of the normal distribution, a random variation range can be given for each pre-given probability (confidence level). Here, for example, the confidence levels of 95.4 % and 99.7 % accompany the limits of random variation $\mu \pm \frac{2 \cdot \sigma}{\sqrt{n}}$ and $\mu \pm \frac{3 \cdot \sigma}{\sqrt{n}}$.

The reversed inference (indirect inference) is likewise possible.

With a probability of 68.3 %, for example, the following applies for the average μ of the population: $\bar{x} - \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + \frac{\sigma}{\sqrt{n}}$, whereby $\bar{x}$ is the average of a sample of n individual values. With the indirect inference, one is dealing with a confidence interval rather than a random variation range.

With the confidence levels of 95.4 % and 99.7 %, μ lies within the respective confidence interval $\bar{x} \pm \frac{2 \cdot \sigma}{\sqrt{n}}$ and $\bar{x} \pm \frac{3 \cdot \sigma}{\sqrt{n}}$.

The confidence levels listed in the following table are used frequently in association with the respective factor u in the expression $\bar{x} \pm u \cdot \frac{\sigma}{\sqrt{n}}$.

P_A	Factor u
90 %	1.64
95 %	1.96
95.4 %	2.0
99 %	2.58
99.7 %	3.0

Let's assume that the average of the population in Example 5.1 has the value $\mu = 7.5\,mm$, and the standard deviation is $\sigma = 0.5\,mm$.

The random variation range of $\bar{x}$ with a 95 % confidence level for a sample with 10 values would then be calculated as

$$\mu \pm u \cdot \frac{\sigma}{\sqrt{n}} = 7.5\,mm \pm 1.96 \cdot \frac{0.5\,mm}{\sqrt{10}} = 7.5 \pm 0.3\,mm\,.$$

A sample with the size $n = 10$ belongs to the population with the characteristics $\mu = 7.5\,mm$ and $\sigma = 0.5\,mm$ with a 95 % confidence level, if the accompanying estimated value $\bar{x}$ lies between 7.2mm and 7.8mm.

The confidence level of 95 % can be interpreted in the following ways:

If such samples were conducted very often with 10 measurement values and the same population, then the accompanying estimated value $\bar{x}$ lay within the limits 7.2mm and 7.8mm in 95% of the cases, and did not do so in 5 % of the cases.

In a reverse manner, in Example 5.1 with unknown μ, an average $\bar{x}$ can be calculated as an estimated value with the assistance of a sample, e.g. $\bar{x} = 7.3$ from 10 values.

Through an indirect inference, one then obtains the statement that μ lies within the confidence interval $7.3\,mm \pm 1.96 \cdot \frac{0.5\,mm}{\sqrt{10}}$ with a confidence level of 95 %.

The more individual values that are brought into play in the estimate, the smaller the confidence interval will be.

The sample size must, then, orient itself to the demanded accuracy of the statement.

It was assumed in this section that the standard deviation σ of the population is known. This is the case, for example, in production when σ of all manufactured parts (population) has been determined.

6 Statistical Tests

6.1 Principle Sequence of a Statistical Test

1. Enumeration of the prerequisites. The individual values of empirically determined data are considered to be the realization of a theoretical random variable. One or more qualities of this random variable are presupposed as known.
2. Formulation of two mutually exclusive statements, a null hypothesis H_0 and an alternative hypothesis H_1. With comparative tests, these hypotheses can refer to two or several random variables (that is, data sets named under 1.).
3. Calculation of a test statistic from the present data.
4. Selection of a significance level α (error probability). A statistical test enables no correct decision with absolute certainty between the null hypothesis and the alternative hypothesis. A small probability must be granted with which the test decision can be false.
5. Comparison of the test statistic with a tabulated percentage point (quantile) accompanying the significance level α determined in 3.

   Test decision: If the test statistic is greater than the percentage point, then the null hypothesis is rejected.

   The alternative hypothesis contains two cases with the two-sided test. Correspondingly, the comparison of the test statistic occurs with a lower or an upper percentage point. The null hypothesis is rejected if the upper percentage point is exceeded or if the lower percentage point is gone below.

Mathematically speaking, the central component of a statistical test is a random variable with a known probability distribution. This random variable can either be the random variable given in 1. itself or an auxiliary quantity constructed through the linking of several random variables named in 1.

As long as the null hypothesis is correct, the test statistic calculated from the data which is, so to speak, a realization of the considered random variable, would assume values in repeated data gaining (sampling) which lie below a percentage point clearly determined by the level of significance in the majority of the cases. An exceeding of the threshold would, then, only occur very rarely.

If this is the case in the sample actually considered, one accepts this not as a possible rare case, but rather concludes from it that the null hypothesis is false.

Depending on the formulation of the alternative hypothesis, an interval can also be given which is limited by an upper and a lower percentage point.

6.2 z Test

As an example of a test and the accompanying test statistic z, the knowledge of Section 5 on the distribution of sample averages with known standard deviation of the population will be of help.

Prerequisites for the z Test:

- the standard deviation σ of the population is known
- the distribution of the population is a normal distribution $N(\mu, \sigma^2)$

We will assume that a sample of size n with an average of $\bar{x}$ is present. The decision is to be made as to whether this sample can originate from a population whose average is $\mu = \mu_0$ ([10] comparison of the expected value with a pre-given value in the case of a normal distribution).

It is assumed as a basis for the test that μ is actually equal to μ_0, that is, that the null hypothesis reads $H_0: \mu = \mu_0$.

If the null hypothesis is false, either $\mu < \mu_0$ or $\mu > \mu_0$.

We will presuppose at this point that it doesn't matter at the moment which of these two cases is concerned in the rejection of the null hypothesis. The alternative hypothesis is therefore:

$H_1: \mu \neq \mu_0$.

In the next step, the test statistic z is determined by using the sample average $\bar{x}$, the assumed average $\mu = \mu_0$ and the known presupposed standard deviation of the average $\sigma_{\bar{x}} = \frac{\sigma}{\sqrt{n}}$:

$$z = \frac{\bar{x} - \mu}{\sigma_{\bar{x}}}.$$

The actual test consists of a comparison of this test statistic z with a lower quantile (percentage point) $z_{\alpha/2}$ and an upper quantile $z_{1-\alpha/2}$ (significance limits).

After the selection of a (small) error probability (significance level) α (e.g. 1 %), $z_{\alpha/2}$ and $z_{1-\alpha/2}$ can be taken from the table of the standard normal distribution for the probability $P_A = 1 - \alpha$ (e.g. $P_A = 99\%$). Here, z corresponds to the quantity u in Table 12.1. For $\alpha = 0.01 = 1\%$, $z_{\alpha/2} = z_{0.005} = -2.58$ and $z_{1-\alpha/2} = z_{0.995} = +2.58$.

One should already determine and also maintain the significance level α before the calculation of the test statistic z. Otherwise, the risk exists of "stealing a glance" at the test statistic z while finding the percentage point in the table and influencing the test outcome through spontaneous modification of α.

Test decision:

If $z < z_{\alpha/2}$ or $z > z_{1-\alpha/2}$, then H_0 is rejected, i.e. the sample cannot originate from a population with $\mu = \mu_0$ with the significance level of α.

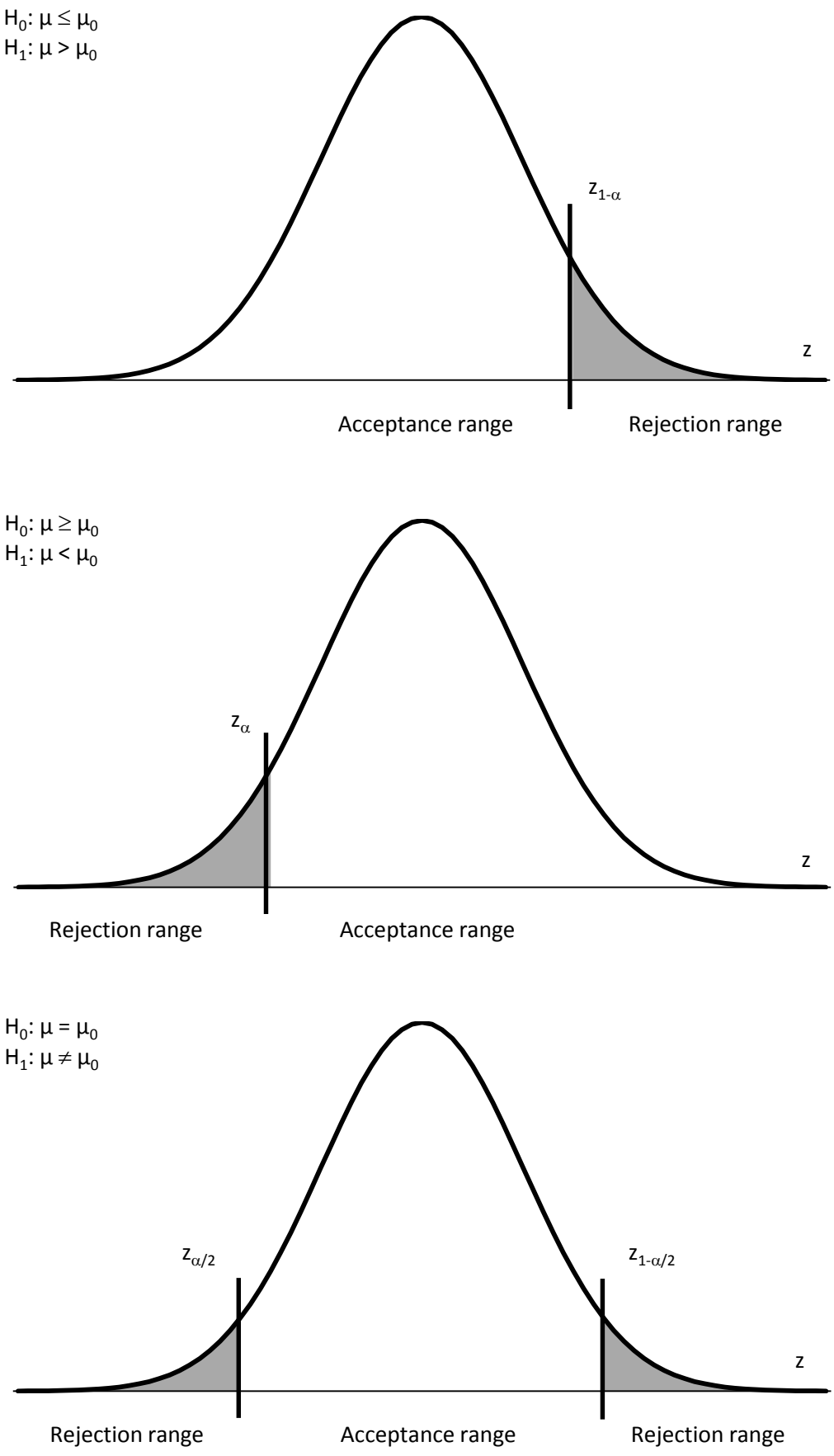


Fig. 6.1: Representation of the acceptance and rejection range with one-sided and two-sided tests

The test is based on the fact that the distribution of the test statistic z is known. It is a standard normal distribution. It is examined as to whether the calculated value z still lies within the random variation range which is given by the probability P_A and by the level of significance $\alpha = 1 - P_A$. Due to the alternative hypothesis $\mu \neq \mu_0$, both external "points" of the standard normal distribution must be taken into consideration with respect to α (Fig. 6.1 bottom). A two-sided question is therefore spoken of in this context. With the one-sided test, the null hypothesis is rejected when $z > z_{1-\alpha}$ (Fig. 6.1 top) or $z < z_{\alpha}$ (Fig. 6.1 center).

EXAMPLE: For $\alpha = 0.01 = 1\%$, $z_{\alpha} = z_{0.01} = -2.33$ and $z_{1-\alpha} = z_{0.99} = +2.33$.

Summary:

The previous example contains all of the essential steps of a statistical test:

1. Formulation of a null hypothesis
2. Calculation of a test statistic whose distribution is known
3. Selection of a level of significance
4. Finding of a quantile (percentage point) in the distribution table
5. for the test statistic
6. Comparison of the test statistic with the quantile; test decision.

6.3 Erroneous Decisions with Statistical Tests

In Section 6.2, a two-sided question was considered, i.e. the null hypothesis $H_0: \mu = \mu_0$ is tested against the alternative hypothesis $H_1: \mu \neq \mu_0$.

With a one-sided question, the null hypothesis reads, for example, $H_0: \mu \leq \mu_0$ and the alternative hypothesis correspondingly reads $H_1: \mu > \mu_0$ (Fig. 6.1 top). The background of this one-sided test could be the question, for example, as to whether the shear strength of a soldered joint as opposed to the average up to now increases (alternative hypothesis is correct) or decreases (alternative hypothesis is false) through the initiation of a process modification.

Two types of erroneous decisions are possible within the scope of this test:

- Type I Error (Error of the first kind): The null hypothesis is rejected falsely. In the named example, this means that one conducts a process modification on the basis of the test outcome, even though this has no effects or possibly even decreases the strength. The accompanying significance level is $\alpha = 1 - P_A$.

- Type II Error (Error of the second kind): The null hypothesis is maintained falsely. This means that one declines a process modification on the basis of the test outcome, even though this would increase the strength considerably. The accompanying significance level is β.

Both erroneous decisions are "uncomfortable". In the first case, one makes an expensive investment unnecessarily, while in the second case a competitive advantage is lost, for example.

Error types, acceptance probabilities and risks with statistical tests:

Accepted facts due to a test outcome	unknown, true facts	
	H_0 is correct ($\mu \leq \mu_0$).	H_1 is correct ($\mu > \mu_0$).
H_0 is correct	Correct decision! $P = 1 - \alpha$	Type II Error $P = \beta$
H_1 is correct	Type I Error $P = \alpha$	Correct decision! $P = 1 - \beta$

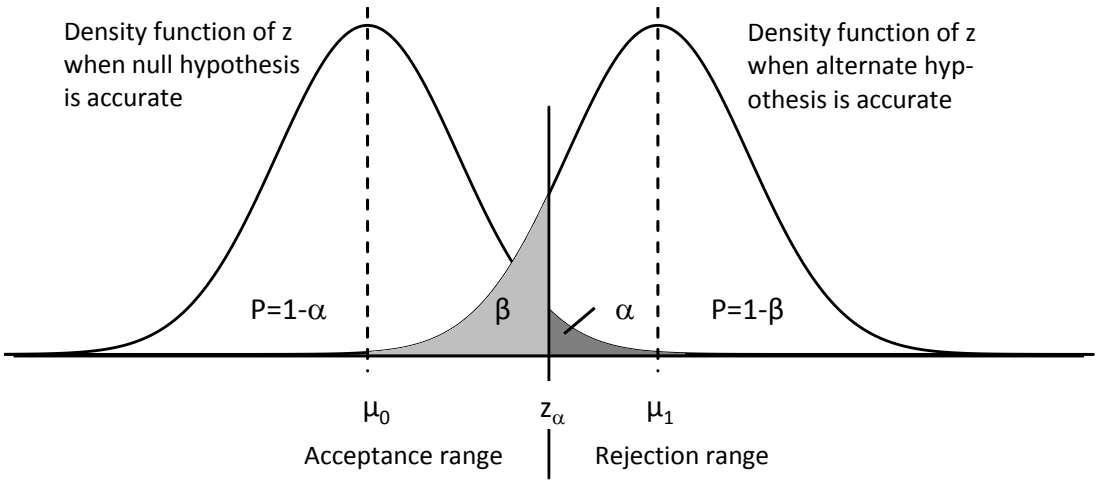


Fig. 6.2: Risks in a statistical test (schematic)

The probability α for type I error is called the significance level in statistical literature. A test outcome is "very significant", then, when α is very small. Since this allocation can easily lead to misunderstandings in general usage, $P = 1 - \alpha$ is usually designated instead as the level of significance and one speaks of "great significance" when, for example, $P = 1 - \alpha > 99\%$.

The determination in this document that a test outcome from $P = 95\%$ is designated as significant and one from $P = 99\%$ is designated as "highly significant" occurs arbitrarily. As long as the results of an erroneous decision make things very difficult in concrete situations (e.g. the endangering of human lives), one will first speak of great significance with numbers which are considerably greater than 99 %.

6.4 Confidence Interval of Averages (σ of the population unknown)

6.4.1 t Distribution

As a rule, the standard deviation σ of a population is unknown. The confidence interval of the average μ of the population can, then, no longer be calculated by way of the standard normally distributed quantity u (compare with Section 4.2), but is rather determined with the assistance of the quantity

$$t = \sqrt{n} \cdot \frac{\bar{x} - \mu}{s}$$

which is subject to the so-called Student's t distribution.

The representation of the density function of the t distribution is very similar to the representation of the normal distribution (see Fig. 6.3). Both are symmetrical, bell-shaped and contain a value range of minus infinity to plus infinity. The smaller the sample size n, the flatter the curve proceeds, i.e. the probability of finding a t value in the runouts is a bit greater in proportion to the standard normal distribution, and the probability of finding a t value in the central area is a bit slighter.

With a large sample size, the t distribution approaches the normal distribution (i.e. the quantity t approaches u).

Since the sample size n is contained in the expression for t, each n has its own t distribution; it is said that the t distribution has f degrees of freedom. This quantity corresponds to the sample size decreased by one: $f = n - 1$.

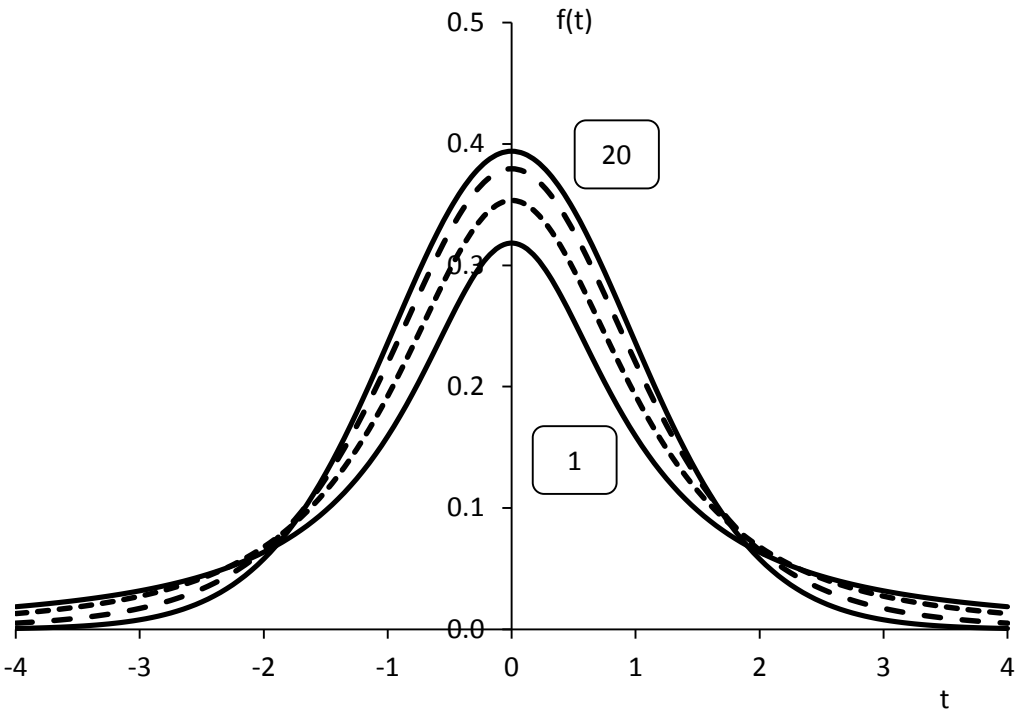


Fig. 6.3: Probability density function of the t distribution for $f = 1, 2, 5, 20$ degrees of freedom. For $f \to \infty$, the t distribution approaches the normal distribution.

The sought after confidence interval for the average μ of the population can be given in the form

$$\bar{x} - t \cdot \frac{s}{\sqrt{n}} \leq \mu \leq \bar{x} + t \cdot \frac{s}{\sqrt{n}}.$$

Here, $\bar{x}$ and s are the average and the empirical standard deviation calculated from the sample of size n.

The quantity t is a value of the tabulated t distribution (Appendix, Table 12.2) for $f = n - 1$ degrees of freedom and the probability P_A. The above expression correspondingly signifies that the unknown average μ lies between $\bar{x} - t_{n-1;\,0.95} \cdot \frac{s}{\sqrt{n}}$ and $\bar{x} + t_{n-1;\,0.95} \cdot \frac{s}{\sqrt{n}}$ with a probability of P_A (of 95 %, for example).

6.4.2 Nomogram for the Determination of the Confidence Interval of μ

The confidence interval of μ can also be determined with the assistance of the nomogram depicted in Fig. 6.4 (confidence level of 95 %).

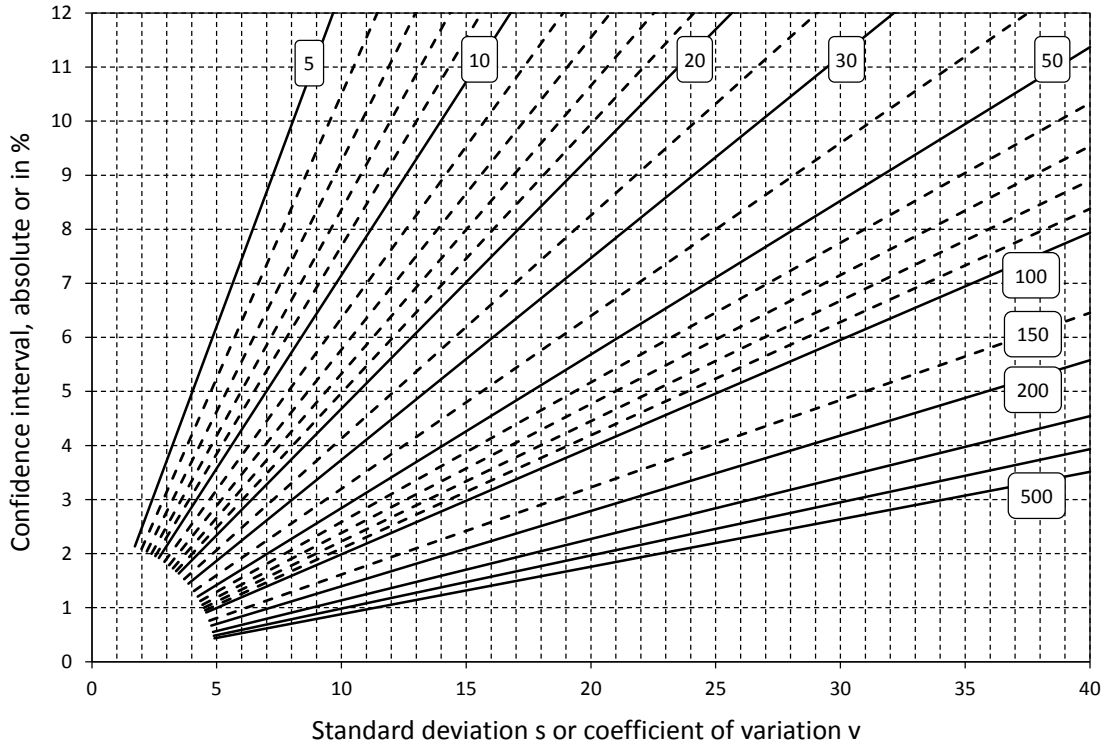


Fig. 6.4: Confidence interval of μ; $P_A = 95\%$ (two-sided)

In Fig. 6.4, the ordinate value $t \cdot \frac{s}{\sqrt{n}}$ (it corresponds to the interval of the confidence limits for μ from sample average $\bar{x}$) is allocated to every abscissa value s dependent on the sample size n.

If one selects the coefficient of variation $v = \frac{s}{\bar{x}} \cdot 100\%$ as abscissa value, one obtains, due to the association $\frac{s \cdot t}{\bar{x} \cdot \sqrt{n}} \cdot 100\% = v \cdot \frac{t}{\sqrt{n}} \cdot 100\%$ as the ordinate value the confidence interval for μ with respect to the average $\bar{x}$. This value is also called the relative error. In both cases, the scale of the abscissa and ordinate axis can be adjusted to the given order of magnitude through equal shifting of the decimal point on both axes.

EXAMPLE 6.1:

From $n = 25$ measurement values of a sample, the average $\bar{x} = 44.0$ and the standard deviation $s = 2.2$ were calculated.

In Fig. 6.4, the ordinate value 9 can be read for the abscissa value 22 and $n = 25$.

If one considers the decimal position, then $t \cdot \frac{s}{\sqrt{n}} = 0.9$.

With a 95 % confidence level, then, $43.1 \leq \mu \leq 44.9$.

The coefficient of variation is: $v = \frac{2.2}{44.0} \cdot 100\,\% = 5\%$.

With abscissa value 5 and $n = 25$, 2.1 can be read as the ordinate value in Fig. 12.

The relative error is thereby ± 2.1 %.

With the assistance of the nomogram in Fig. 6.4, the sample size can also be determined which is necessary in order to remain within a pre-given relative confidence interval.

EXAMPLE 6.2:

A sample average of $\bar{x} = 1.2$ and a standard deviation of $s = 0.029$ are expected.

The coefficient of variation is hereby $v = 2.4\%$.

The sample size with a permitted relative error of 0.5 % at a 95 % confidence level is sought after.

The line for $n = 90$ is drawn by way of the points accompanying the abscissa value of 24 and the ordinate value of 5 in the coordinate field. A sample of 90 measurement values is hereby necessary.

6.5 Random Variation Range and Confidence Interval of the Standard Deviation

The inductive statistics also enable the calculation of the random variation range of the empirical standard deviation s and the confidence interval of the standard deviation σ of the population. This happens with the assistance of the χ^2 function (chi-squared function), from which the given values B_{lower} and B_{upper} were calculated in Table 6.1.

		$P_A = 90\,\%$		$P_A = 95\,\%$		$P_A = 99\,\%$	
$f = n - 1$	n	B_{lower}	B_{upper}	B_{lower}	B_{upper}	B_{lower}	B_{upper}
1	2	0.06	1.96	0.03	2.24	0.01	2.81
2	3	0.24	1.73	0.16	1.92	0.07	2.30
3	4	0.34	1.61	0.27	1.76	0.16	2.07
4	5	0.42	1.54	0.35	1.67	0.23	1.93
5	6	0.48	1.49	0.41	1.60	0.29	1.83
6	7	0.52	1.45	0.45	1.55	0.34	1.76
7	8	0.56	1.42	0.49	1.51	0.38	1.70
8	9	0.58	1.39	0.52	1.48	0.41	1.66
9	10	0.61	1.37	0.55	1.45	0.44	1.62
11	12	0.64	1.34	0.59	1.41	0.49	1.56
13	14	0.67	1.31	0.62	1.38	0.52	1.52
15	16	0.70	1.29	0.65	1.35	0.55	1.48
17	18	0.71	1.27	0.67	1.33	0.58	1.45
19	20	0.73	1.26	0.68	1.31	0.60	1.42
24	25	0.76	1.23	0.72	1.28	0.64	1.38
29	30	0.78	1.21	0.74	1.26	0.67	1.34
34	35	0.80	1.20	0.76	1.24	0.70	1.32
39	40	0.81	1.18	0.78	1.22	0.72	1.30
49	50	0.83	1.16	0.80	1.20	0.75	1.26
59	60	0.85	1.15	0.82	1.18	0.77	1.24
119	120	0.89	1.11	0.88	1.13	0.84	1.17

Table 6.1: Factors for the calculation of the random variation range of s and the confidence interval of σ with confidence levels (two-sided) of $P_A = 90\%$, $P_A = 95\%$ and $P_A = 99\%$. f is the number of the degrees of freedom, and n is the sample size.

When the standard deviation of the population is known or when a hypothetical value can be assumed for it, it can be calculated in which random variation range the standard deviation s of a sample of size n with the pre-given confidence level must lie so that it can belong to this population.

The random variation range for s is given by

$$B_{lower} \cdot \sigma < s < B_{upper} \cdot \sigma .$$

If σ is unknown and one wants to know in which range the true σ lies around an estimated value s with a pre-given confidence level, then the confidence interval for σ is calculated from the estimated value s and the values from Table 6.1 in the following manner:

$$\frac{1}{B_{upper}} \cdot s < \sigma < \frac{1}{B_{lower}} \cdot s .$$

EXAMPLE 6.3:

From the $n = 40$ individual values of a sample, the average $\bar{x} = 100$ and the standard deviation $s = 24$ were calculated.

The following then applies with a confidence level of 95 %:

$$\frac{1}{1.22} \cdot 24 < \sigma < \frac{1}{0.78} \cdot 24 \quad \text{that is} \quad 20 < \sigma < 31 .$$

Unlike with the confidence limits for the average, the intervals of the confidence limits here are not of equal size to the estimated value.

6.6 Test for Randomness

With several tosses of a coin, one expects that the toss outcome “heads” (H) and “tails” (T) will change in an irregular manner. One would particularly look at an outcome sequence like “... T H H H H H H H T T H ...” as unusual. With the coin toss, the probability for each outcome (“H” or “T”) is equal at 0.5.

Due to the independence of the individual tosses, the probability of the occurrence of seven “H” tosses is $(0.5)^7 \approx 0.8\%$.

Such a sequence of identical symbols is called iteration. The outcome sequence “T H H H T H H H T T H” consists of the 6 iterations (T), (H H H), (T), (H H H H), (T T) and (H), that is, 3 iterations of “H” and 3 iterations of “T”.

If one gives a number n of tosses, he expects that the number of iterations is not too great, but also not too small. In the first case, “H” and “T” would change too frequently (too regularly), and in the second case, conversely, too rarely.

The number r of the iterations of the symbol “H” or the symbol “T” in a sequence of n coin tosses is a random quantity. The randomness of a sequence of toss outcomes can be tested with the assistance of this random quantity r.

If the total number of the individual outcomes “H” is equal to n_1 and the total number of individual outcomes “T” is equal to n_2, then a range can be given in which r lies with the probability $P_A = 1 - \alpha$ (compare with [10]).

In the two-sided test, the null hypothesis H_0: “the outcome is random” is rejected when $r_{n_1;n_2;\alpha/2} < r$ or $r > r_{n_1;n_2;1-\alpha/2}$.

This test can be applied to a sequence of measurement values if one converts the individual values x_i $(i = 1, 2, \ldots, n)$ into a sequence of plus and minus signs by way of the following rule ($\tilde{x}$ is the median of the measurement series):

- Replace x_i with “+” if $x_i > \tilde{x}$ and
- replace x_i with “-” if $x_i \leq \tilde{x}$.

The test statistic in this so-called run test is the number r of the iteration of the symbols "+" and "-". A detailed description of this test and the accompanying table of critical values (upper and lower limits of r) can be found, for example, in [10] or [11].

In the similar test according to Wallis and Moore, the sequence of plus and minus signs is formed by respectively calculating the difference between two values which follow one another. This test is also described in the given literature.

A test for randomness can be helpful if doubt exists as to the independence of measurement values which follow one another. This is conceivable, for example, in association with a process data analysis or a machine capability study.

The test responds, for example, when the data set contains one or more "concentrations" of similar values (batch discontinuities), or when a tendency or periodicity is present. Such facts, however, should also be recognizable in this case on the basis of an original value diagram (Section 2.1).

6.7 Outlier Test According to Grubbs

The Grubbs test presupposes that a normally distributed population is present whose average and standard deviation are unknown.

First of all, the empirical average $\bar{x}$ and the standard deviation s are calculated from all n sample values.

If the smallest sample value x_{min} is suspected as an outlier, the test statistic $\frac{\bar{x}-x_{min}}{s}$ is calculated; conversely, if the greatest sample quantity x_{max} is suspected as an outlier, the test statistic $\frac{x_{max}-\bar{x}}{s}$ is formed.

The critical value belonging to the confidence level $P_A = 1-\alpha$ is taken from Table 12.5 and is compared with the test statistic. If the test statistic is greater than the critical value, then a real outlier is present with the probability P_A.

EXAMPLE 6.4:

In the determination of the strength of 15 pieces of wire, the following breaking loads (N) were measured: 76, 84, 82, 88, 89, 83, 72, 70, 82, 54, 82, 76, 88, 87, 78

The value 54 N is suspected as an outlier.

From the $n=14$ values (including the value suspected as an outlier), the following are produced:

the average $\bar{x}=79.4\,N$,

the standard deviation $s=9.12\,N$.

In Table 12.5, the critical value 2.409 is found for $n=14$ and $P_A=95\,\%$.

Because $\frac{\bar{x}-x_{min}}{s}=\frac{79.4-54}{9.12}=2.785>2.409$, *the test result reads:*

The measurement value 54 N is a real outlier with $P_A=97.5\,\%$.

REMARK:

In the inspection of the prerequisite (normal distribution), one can record the individual values with the assistance of median ranks in a probability plot. In this example, it is already visible that the points can be approximated quite well with a line with the exception of the point accompanying the outlier value.

7 Comparing Two Measurement Series

7.1 F Test

In the assessment of two measurement series which, for example, were conducted before or after a procedure modification, a clear difference between both of the empirical variances s_1^2 and s_2^2 can be produced.

The question as to whether this difference is accidental or whether it can be led back to the procedure modification can be answered with the assistance of the F test.

First, the proportion of both variances is calculated:

$$F = \frac{s_1^2}{s_2^2}.$$

s_1^2 is the greater of the two variances and above the fraction line. n_1 is the sample size accompanying s_1^2.

Finally, the test statistic F is compared with the table values F(95%) and F(99%) with reference to $f_1 = n_1 - 1$ and $f_2 = n_2 - 1$ degrees of freedom (Table 12.3, two-sided).

Test results:

If ...	... is the statistical difference between both variances
$F \geq F(99\%)$	highly significant
$F(99\%) > F \geq F(95\%)$	significant
$F(95\%) > F$	insignificant

EXAMPLE 7.1:

Within the scope of a comparison examination, breaking loads (N) of pieces of wire were determined from two differently handled batches.

The statistical characteristics of the measurement series are:

Batch A: $\bar{x}_A = 82.3N$ $s_A^2 = 18\,N^2$ $n_A = 10$

Batch B: $\bar{x}_B = 90.0N$ $s_B^2 = 128\,N^2$ $n_B = 12$

While the averages are similar at first glance, the deviation of the variances appears quite great.

One calculates: $F = \frac{s_1^2}{s_2^2} = \frac{128N^2}{18N^2} = 7.1$.

Degrees of freedom: $f_1 = n_1 - 1 = 11$ *and* $f_2 = n_2 - 1 = 9$.

In Table 12.3 (two-sided) of the F distribution in the appendix one finds for $f_1 = 10$ *and* $f_2 = 9$ *degrees of freedom the quantile* $F(99\%) = 6.42$. *Since the table contains no column for the 1st number of degrees of freedom* $f_1 = 11$, *the next smaller given number (w.r.t.* $f_1 = 10$*) can be selected problem-free as the 99 % point. This means merely a negligible shifting of the significance level.*

Because $F \geq F(99\%)$, *the test decision then reads:*

The difference between both of the variances is highly significant.

The F test in the form represented here tests the null hypothesis $\mu_1 = \mu_2$ against the alternative $\mu_1 \neq \mu_2$. A two-sided question is then present.

Table 12.3 (two-sided) in the appendix gives the two-sided 95 % and 99 % points of the F distribution in accordance with the two-sided question. They correspond to the one-sided 97.5 % and 99.5 % points (percentage points; quantiles).

7.2 t Test

7.2.1 t Test for Samples with Equal Variances

With the assistance of a t test, it should be decided as to whether the averages $\bar{x}_1$ and $\bar{x}_2$ of two samples of size n_1 and n_2 are significantly different, or whether both of the samples originate from a mutual population.

The manner of functioning of the t test can be explained on the basis of the following thought experiment.

We take two samples with size $n = n_1 = n_2$ from a normally distributed population N(μ, σ), calculate the averages $\bar{x}_1$ and $\bar{x}_2$ and the standard deviations s_1 and s_2 (and the variances s_1^2 and s_2^2) and finally determine the value

$$t = \sqrt{n} \cdot \frac{\left|\bar{x}_1 - \bar{x}_2\right|}{\sqrt{s_1^2 + s_2^2}}.$$

t can assume values between 0 and $+\infty$. If we repeat this process very often, we will expect to find mostly small values close to zero and very rarely to find large values.

t possesses a probability density function whose graph is conveyed in Fig. 7.1 for the case $n = 10$.

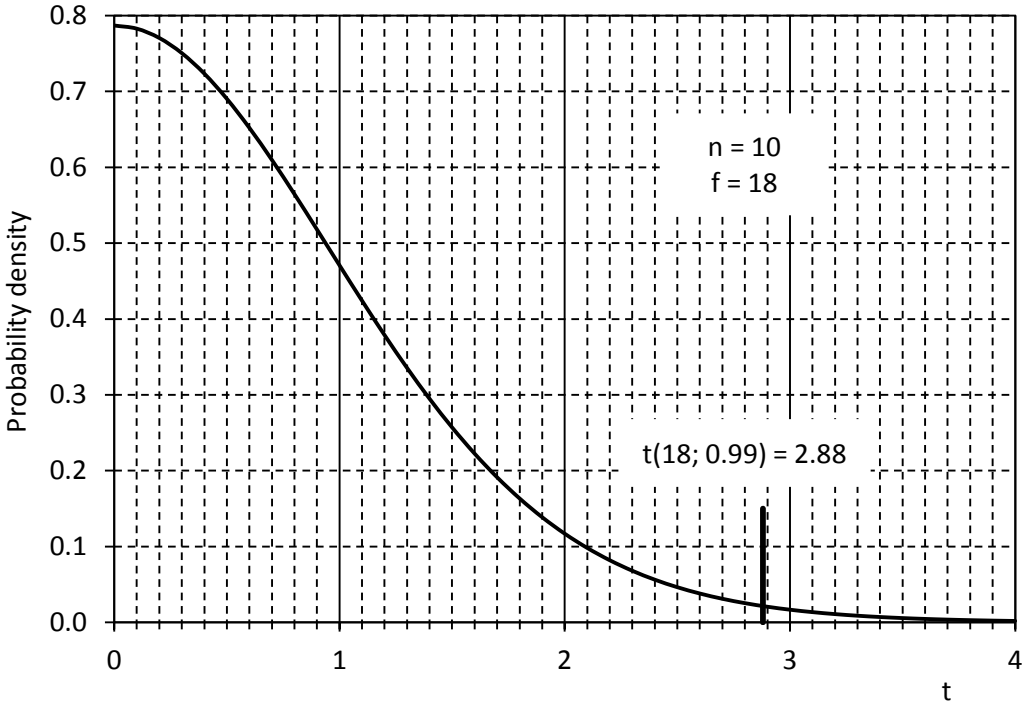


Fig. 7.1

The upper limit of the 99 % random variation range (percentage point; quantile) of t is in this case $t_{18;\,0.99} = 2.88$, i.e. only in 1 % of all cases can a random value greater than 2.88 be produced. If this case should actually occur during the comparison of two samples with $n = 10$, one would conclude that the samples cannot originate from the same population.

The quantiles of the t distribution are tabulated for various significance levels dependent on the degrees of freedom $f = 2 \cdot (n - 1)$ (Table 12.2). The procedure during the t test is based on the correlation represented above.

It should be decided as to whether the arithmetic mean of two present measurement series (each of size n) can or cannot belong to one and the same population. It is then assumed as the so-called null hypothesis that the averages of the respective accompanying population are equal.

From the averages $\bar{x}_1$ and $\bar{x}_2$ and the variances s_1^2 and s_2^2, the following test statistic is then calculated:

$$t = \sqrt{n} \cdot \frac{\left| \bar{x}_1 - \bar{x}_2 \right|}{\sqrt{s_1^2 + s_2^2}} \qquad \text{for } n_1 = n_2 = n$$

Through comparison of the test statistic t with the table values t(95%) and t(99%) for $f = n_1 + n_2 - 2$ degrees of freedom (Table 12.2, two-sided), one finally obtains the

Test outcome:

If ...	... is the statistical difference between both averages
$t \geq$ t(99%)	highly significant
t(99%) $>$ t $\geq$ t(95%)	significant
t(95%) $>$ t	insignificant

The t test in the form represented here tests the null hypothesis $\mu_1 = \mu_2$ against the alternative $\mu_1 \neq \mu_2$. A two-sided question is then present. For this reason, the amount of the average difference stands for t in the expression.

t can consequently only assume values ≥ 0, so that the distribution shown in Fig. 7.1 is produced.

Table 12.2 (two-sided) in the appendix gives the two-sided 95 %, 99 %, and 99.9 % points of the t distribution, among others, in accordance with the two-sided question. They correspond to the one-sided 97.5 %, 99.5 % and 99.95 % points (percentage points; quantiles).

The expression for the test statistic t is only applicable in this simple form when the variances of the population and the sample sizes are presupposed respectively as being equal ($\sigma_1^2 = \sigma_2^2$ and $n_1 = n_2 = n$). The prerequisite of equal variances can be inspected with the assistance of an F test (compare with 7.1).

In the case $n_1 \neq n_2$, the expression for t, in using the auxiliary quantity,

$$s_D = \sqrt{\frac{(n_1-1)\cdot s_1^2+(n_2-1)\cdot s_2^2}{n_1+n_2-2}\cdot\frac{n_1+n_2}{n_1\cdot n_2}} \quad \text{obtains the form} \quad t = \frac{\left|\bar{x}_1-\bar{x}_2\right|}{s_D}$$

The test decision occurs as above; however, due to $n_1 \neq n_2$ an alternate number of degrees of freedom must be calculated for t and must be taken into consideration in finding the table values (compare with 7.2.2).

7.2.2 t Test for Samples with Unequal Variances

If the variances of the population from which the samples originate are not necessarily equal, the calculation of the test statistic t occurs with the assistance of the expression:

$$s_D = \sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}} \quad \text{in the form} \quad t = \frac{\left|\bar{x}_1-\bar{x}_2\right|}{s_D}.$$

An alternate number of degrees of freedom f is to be calculated according to the following formula [14]:

$$f = \frac{s_D^4}{\frac{\left(\frac{s_1^2}{n_1}\right)^2}{n_1-1}+\frac{\left(\frac{s_2^2}{n_2}\right)^2}{n_2-1}}. \qquad \text{(round f down to a whole number!)}$$

The rather laborious calculation of f can frequently be replaced by an estimate.

If ...	..., then select
a) $\frac{s_1^2}{n_1} \approx \frac{s_2^2}{n_2}$	$f = n_1+n_2-2$
b) $\frac{s_1^2}{n_1} >> \frac{s_2^2}{n_2}$	$f = n_1-1$
c) $\frac{s_1^2}{n_1} << \frac{s_2^2}{n_2}$	$f = n_2-1$

Through comparison of the test statistic t with the table values t(95%) and t(99%) for f degrees of freedom (Table 12.2, two-sided) the following is produced:

Test outcome:

If ...	... is the statistical difference between both averages
$t \geq t(99\%)$	highly significant
$t(99\%) > t \geq t(95\%)$	significant
$t(95\%) > t$	insignificant

EXAMPLE 7.2:

In the example 7.1 for the F test, the following statistical characteristics were determined:

$\bar{x}_1 = 82.3\,N \qquad s_1^2 = 18\,N^2 \qquad n_1 = 10$

$\bar{x}_2 = 90.0\,N \qquad s_2^2 = 128\,N^2 \qquad n_2 = 12$

Here, $\left|\bar{x}_1 - \bar{x}_2\right| = 7.7\,N$ *and* $s_D = \sqrt{\frac{18\,N^2}{10} + \frac{128\,N^2}{12}} = 3.5\,N$ *are produced,*

and consequently $t = \frac{7.7\,N}{3.5\,N} = 2.2$.

$$f = \frac{3.5^4}{\frac{\left(\frac{18}{10}\right)^2}{10-1} + \frac{\left(\frac{128}{12}\right)^2}{12-1}} \approx 14$$

For $f = 14$ *degrees of freedom, one finds the value* $t(95\,\%) = 2.145$ *in Table 12.2 (two-sided).*

Because $t(99\,\%) > t \geq t(95\,\%)$, *the difference between the averages is significant with a significance level of* $\alpha \leq 5\,\%$.

COMMENT:

Since the null hypothesis $H_0 : \mu_1 = \mu_2$ *is to be tested against the alternative* $H_1 : \mu_1 \neq \mu_2$, *we are dealing with a two-sided question. The test statistic* t *is to therefore be taken from Table 12.2 (two-sided).*

In statistics textbooks and table volumes, usually only the table of t values is conveyed for the one-sided question. If such a table is to be used, it is to be observed that the confidence level $P_A = 95\%$ *(one-sided) corresponds to the confidence level* $P_A = 90\,\%$ *(two-sided).*

The value of the test statistic $t(95\,\%) = 2.145$ *(compare with example) can be found, then, in Table 12.2 (two-sided) in the 95 % column, as expected, but in column 97.5 % in common tables of the t distribution (one-sided) (each for* $f = 14$ *degrees of freedom).*

7.3 Difference Test

If the selection of components is measured several times, varying measurement series can occur due to modified basic conditions (e.g. measurement system, procedure, location, time, inspector). The question as to whether the difference between two measurement series is of random nature or not can be answered with the assistance of the difference test (compare with [17], paired comparison with the t test [10]).

The prerequisite with this test is that both measurement series are of equal size ($n_1 = n_2 = n$) and that a paired allocation of the measurement values of both series is possible.

A tabular arrangement of the measurement outcomes eases the implementation of the test. It is to be observed that the differences $d_i = x_i - y_i$ are given in consideration of the sign.

Object No.	Measurement series 1	Measurement series 2	Difference d_i	d_i^2
1	x_1	y_1	$d_1 = x_1 - y_1$	d_1^2
2	x_2	y_2	$d_2 = x_2 - y_2$	d_2^2
3	x_3	y_3	$d_3 = x_3 - y_3$	d_3^2
...	...	...	...	...
...	...	...	...	...
n	x_n	y_n	$d_n = x_n - y_n$	d_n^2

Table 7.1: Designations with the difference test

Procedure for the difference test:

1. Creation of the table (as above). Calculation of the differences d_i.
2. Calculation of the average $\bar{d}$ and the standard deviation s_d of the differences d_i (pocket calculator).
3. Calculation of the test statistic $t = \sqrt{n} \cdot \frac{|\bar{d}|}{s_d}$.
4. Comparison of the test statistic t with the table values t(95%) and t(99%) for $f = n - 1$ degrees of freedom (Table 12.2, two-sided).

Test outcome:

If ...	... is the statistical difference between both measurement series
$t \geq$ t(99%)	highly significant
t(99%) $> t \geq$ t(95%)	significant
t(95%) $> t$	insignificant

8 Linking of Random Variables

Within the scope of the statistical process control (SPC), samples of parts are taken from a production process in regular intervals. The interested characteristic (e.g. diameter) is measured with each part of the sample and the outcome is recorded on a quality control chart. On the basis of a completed control chart, an estimated value $\hat{\sigma}$ for the standard deviation σ of the population of all parts manufactured within the observation time period can be determined. Since σ is a measure of the quality of the process considered, among others, the question is frequently discussed as to how the measurement uncertainties of the measuring instrument used affect the process standard deviation.

If we disregard a systematic measurement error, a measured characteristic value z is combined with the "true" value x of the characteristic and a random measurement error y: $z = x + y$.

Let's assume that a process in control with a stable average position and stable variation is being considered. The standard deviation σ_F is a measure for the variability of the production, and σ_M is a measure for the variability of the measurement. The standard deviation of the measured characteristic value is then given by $\sigma_z = \sqrt{\sigma_F^2 + \sigma_M^2}$.

It is clear that the random measurement error will lead to an increase of the standard deviation of the measured characteristic value in any case, thereby decreasing the product quality. In a purely formal manner, the eclipsing of production variation and measurement variation corresponds to the addition of two independent random quantities: $Z = X + Y$.

It is generally applicable for this case that the average of Z is produced by the addition of the averages of X and Y: $\mu_z = \mu_x + \mu_y$.

The variance of Z is obtained through the addition of the variances of X and Y: $\sigma_z^2 = \sigma_x^2 + \sigma_y^2$.

The calculation rules for μ and σ for several simple linkings (functions) of two independent (correlation coefficient $\rho_{xy} = 0$) random quantities X (average μ_x, standard deviation σ_x) and Y (average μ_y, standard deviation σ_y) are listed in the following table (compare with [10]).

Function	Average	Variance
$Z = X + Y$	$\mu_z = \mu_x + \mu_y$	$\sigma_z^2 = \sigma_x^2 + \sigma_y^2$
$Z = X - Y$	$\mu_z = \mu_x - \mu_y$	$\sigma_z^2 = \sigma_x^2 + \sigma_y^2$
$Z = X \cdot Y$	$\mu_z = \mu_x \cdot \mu_y$	$\sigma_z^2 \approx \mu_x^2 \cdot \sigma_y^2 + \mu_y^2 \cdot \sigma_x^2$
$Z = \frac{X}{Y}$	$\mu_z \approx \frac{\mu_x}{\mu_y} \cdot \left(1 + \frac{\sigma_y^2}{\mu_y^2}\right)$	$\sigma_z^2 \approx \frac{\mu_x^2}{\mu_y^2} \cdot \left(\frac{\sigma_x^2}{\mu_x^2} + \frac{\sigma_y^2}{\mu_y^2}\right)$

Table 8.1: Calculation rules for linkings of independent random quantities

COMMENTS:

The same calculation rules apply for $\bar{x}$ (or s) as for μ (or σ).

These calculation rules apply only under the prerequisite that the individual quantities X and Y are independent of one another, i.e. that the special value of X does not influence the random selection of Y.

It is expressly emphasized that the variance of the quantity resulting from both the addition as well as the subtraction of two random quantities is calculated by the addition of both individual variances (compare with Table 8.1).

EXAMPLE 8.1: Addition of two lengths

$\bar{x}_a = 30\ mm$, $s_a = 0.02\ mm$

$\bar{x}_b = 50\ mm$, $s_a = 0.02\ mm$

$\bar{x}_c = \bar{x}_a + \bar{x}_b = 80\ mm$

$s_c^2 = s_a^2 + s_b^2$

$s_c = \sqrt{(0.02\ mm)^2 + (0.02\ mm)^2} = 0.028\ mm$

9 Decomposition of Mixed Distributions

With some empirically gained distributions, the suspicion exists that a mixture of two or more distributions is present (especially when the frequency distribution is bimodal or multimodal). Here, only the case of the mixed distribution of two normal distributions will be dealt with.

This occurs in technology, for example, when the partial collectives of two different production processes or production lines are mixed with one another.

If information is to be gained on the averages and standard deviations of the affected partial collectives, a decomposition of the mixed distribution must be conducted in two normally distributed partial collectives with the corresponding characteristics $\bar{x}$ and s.

The procedure will be explained in an example. Fig. 9.1 shows the control chart with probability plot for the example.

The decomposition of mixed distributions is only sensible if a greater number of values is available (at least $n = 100$ to $n = 250$).

The representation of the cumulative frequency appears in the probability plot as a curve which curves partially downwards and partially upwards, indicating a mixed distribution.

The columns 3-5 of Table 9.1 contain the values n_j, G_j and H_j taken from the control chart. The class limits are listed in column 2.

The relative frequencies h_j can be calculated from the cumulative relative frequencies H_j (column 6). The h_j are finally plotted against the accompanying classes. The frequency distribution of the mixed collective is obtained in this manner.

The left partial collective I should now be determined. For this, the relative frequency h_j is transferred to Table 9.1 (column 6, starting with h_1 to the first maximum of the h_j) into column 7. In our example, h_j attains its first maximum in the 9th class ($j = 9$).

Corresponding to the incline up to this first maximum value, the same values are then assumed in reverse sequence into the following classes; that is, the frequency of class 8 into class 10, etc.

The relative frequency values $h_j'(I)$ gained in this manner belong to partial collective I and can be recorded for illustration in Fig. 9.2.

The relative frequency values $h_j'(II)$ of partial collective II are produced by the difference from the relative frequencies h_j of the total distribution (column 6) and the relative frequencies $h_j'(I)$ (column 7). They are recorded in column 8 of Table 9.1.

These values are also finally drawn into Fig. 9.2. The given mixed distribution is hereby subdivided into two symmetrically formed partial collectives I and II.

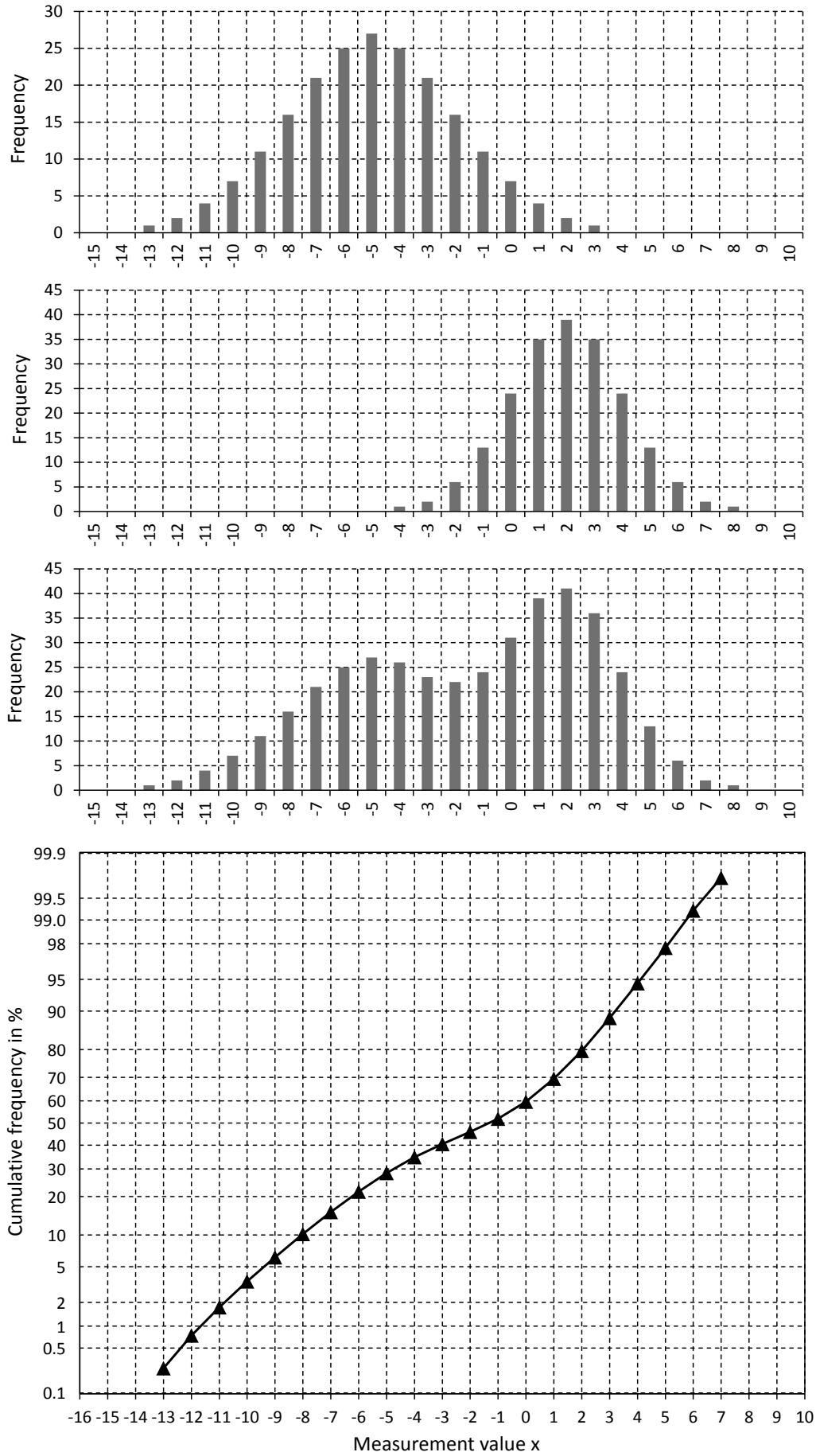


Fig. 9.1: Representation of the mixed distribution from the example in this section

1	2a	2b	3	4	5	6	7	8	9	10	11	12
Class No.	Lower class limit	Upper class limit	Absolute frequency n_j	Cumul. frequency G_j	Cumul. rel. freq. H_j	Relative Frequency h_j	Not stand. rel. freq. $h_j'(I)$	Not stand. rel. freq. $h_j'(II)$	Standard. rel. freq. $h_j(I)$	Standard. rel. freq. $h_j(II)$	Cumul. rel. freq. $H_j(I)$	Cumul. rel. freq. $H_j(II)$
1	-13.5	-12.5	1	1	0.25	0.25	0.25		0.5		0.5	
2	-12.5	-11.5	2	3	0.75	0.50	0.50		1.0		1.5	
3	-11.5	-10.5	4	7	1.74	0.99	0.99		1.98		3.48	
4	-10.5	-9.5	7	14	3.5	1.76	1.76		3.52		7.0	
5	-9.5	-8.5	11	25	6.2	2.7	2.7		5.4		12.4	
6	-8.5	-7.5	16	41	10.2	4.0	4.0		8.0		20.4	
7	-7.5	-6.5	21	62	15.4	5.2	5.2		10.4		30.8	
8	-6.5	-5.5	25	87	21.6	6.2	6.2		12.4		43.2	
9	-5.5	-4.5	27	114	28.4	6.8	6.8		13.6		56.8	
10	-4.5	-3.5	26	140	34.8	6.4	6.2	0.2	12.4	0.4	69.2	0.4
11	-3.5	-2.5	23	163	40.6	5.8	5.2	0.6	10.4	1.2	79.6	1.6
12	-2.5	-1.5	22	185	46.0	5.4	4.0	1.4	8.0	2.8	87.6	4.4
13	-1.5	-0.5	24	209	52.0	6.0	2.7	3.3	5.4	6.6	93.0	11.0
14	-0.5	0.5	31	240	59.7	7.7	1.76	5.94	3.52	11.88	96.52	22.88
15	0.5	1.5	39	279	69.4	9.7	0.99	8.71	1.98	17.42	98.5	40.3
16	1.5	2.5	41	320	79.6	10.2	0.50	9.7	1.0	19.4	99.5	59.7
17	2.5	3.5	36	356	88.6	9.0	0.25	8.75	0.5	17.5	100	77.2
18	3.5	4.5	24	380	94.5	5.9		5.9		11.8		89.0
19	4.5	5.5	13	393	97.8	3.3		3.3		6.6		95.6
20	5.5	6.5	6	399	99.3	1.5		1.5		3.0		98.6
21	6.5	7.5	2	401	99.8	0.5		0.5		1.0		99.6
22	7.5	8.5	1	402	100	0.2		0.25		0.4		100
Σ	-	-	402	-	-	100	50.0	50.0	100	100	-	-

Table 9.1: Value table for the explanation of the decomposition process

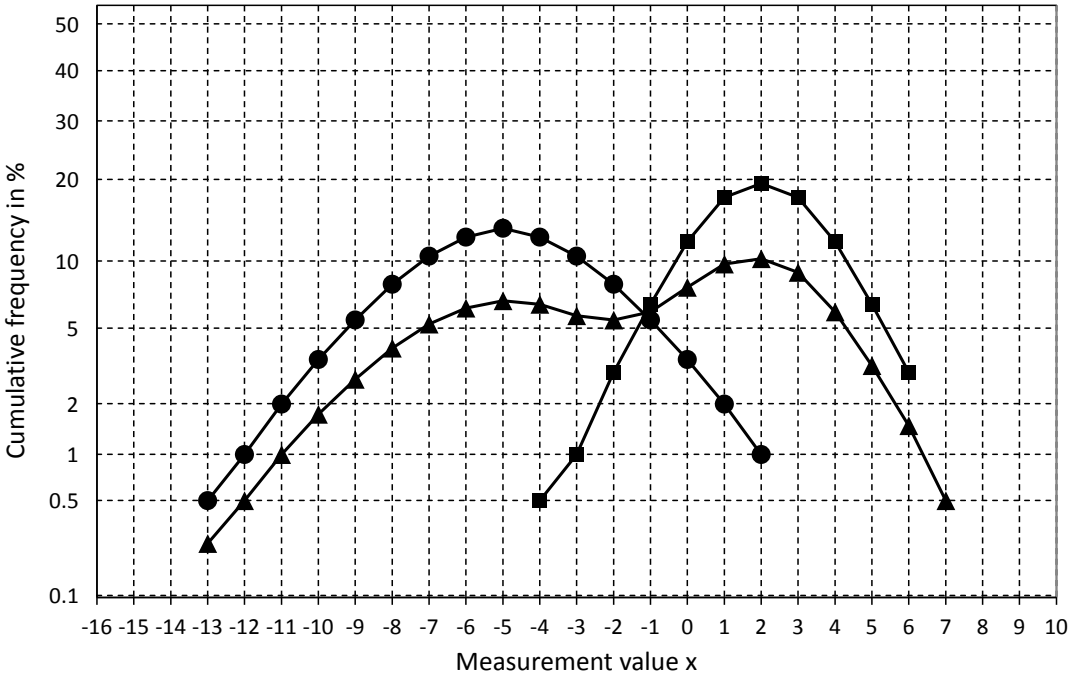


Fig. 9.2: Frequency distribution of the mixed collective

COMMENT:

Due to the procedure described here, the distribution of the left partial collective is inevitably symmetrical. If the population of the left collective were in actuality subject to a distribution with positive slope, values would be falsely allotted to the right partial collective from the transition range between the averages!

Since the relative frequencies $h'_j(I)$ and $h'_j(II)$ were determined through a decomposition, their respective sums will not be the value one. It is necessary, then, to still standardize the frequency distributions of both partial collectives.

In our example, $\sum_{j=1}^{17} h'_j(I) = 50$ *and* $\sum_{j=10}^{22} h'_j(II) = 50$.

The standardization to one is attained in this case by the multiplication of the relative frequencies $h'_j(I)$ *and* $h'_j(II)$ *with the factor 2.*

In general, the standardization equation reads as follows:

$$h_j(I) = \frac{100 \cdot h'_j(I)}{\sum h'_j(I)} \quad \text{and} \quad h_j(II) = \frac{100 \cdot h'_j(II)}{\sum h'_j(II)} \qquad \text{(all statements in percent!)}$$

COMMENT:

The summation indices were consciously disregarded. It is useful to renumber the values in both columns 7 and 8, since otherwise confusion could ensue with respect to the indexing.

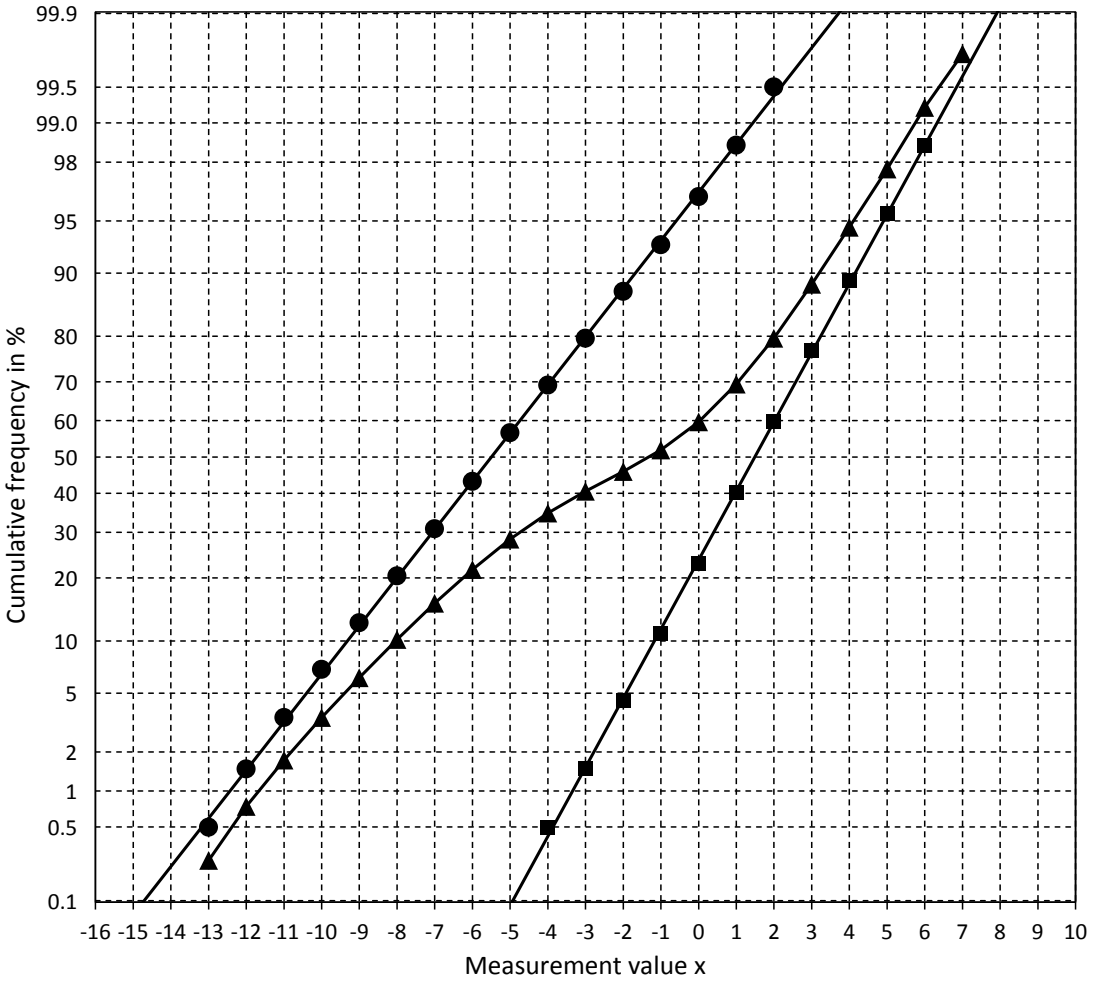


Fig. 9.3: Cumulative frequency curves of the mixed collective and the partial collectives in a probability plot

The cumulative relative frequencies $H_j(I)$ and $H_j(II)$ can finally be calculated from the determined values $h_j(I)$ and $h_j(II)$ (columns 11 and 12). In the example, the dot sequences appearing in the probability plot can be approximated quite well by two separated compensating lines (Fig. 9.3). The mixed collective considered was produced in this case from two normally distributed partial collectives.

The characteristics for both of the partial collectives were read from the probability plot in the usual manner.

Partial collective I:	$\bar{x}_1 = -5.0$	$s_1 = 2.9$
Partial collective II	$\bar{x}_2 = 1.9$	$s_2 = 2.1$

The decomposition of mixed distributions which are composed of more than two populations occurs in an analogous manner.

10 Law of Error Propagation

The Gaussian law of error propagation describes how the measurement errors of several independent measurands x_i affect a target quantity z, which is calculated in accordance with a functional correlation $z = f(x_1, x_2, \ldots, x_k)$:

$$s_z^2 = \left(\frac{\partial z}{\partial x_1}\right)^2 \cdot s_1^2 + \left(\frac{\partial z}{\partial x_2}\right)^2 \cdot s_2^2 + \ldots + \left(\frac{\partial z}{\partial x_k}\right)^2 \cdot s_k^2 .$$

z is, then, generally only an indirectly measurable quantity. The accuracy with which z can be given depends on the accuracy of the measurands x_i. In general, the average $\bar{x}_i$ and the standard deviation s_i of a sequence of measurements are given for each respective x_i.

Application example:

1. The area $F(a,b) = a \cdot b$ of a rectangle is determined by multiply measuring the respective side lengths a and b and multiplying the averages $\bar{a}$ and $\bar{b}$ with each other. The average measurement outcome for F is $\bar{F} = \bar{a} \cdot \bar{b}$. The standard deviation of F is $s_F = \sqrt{\bar{b}^2 \cdot s_a^2 + \bar{a}^2 \cdot s_b^2}$.

2. The law of error propagation can be represented very easily when the function f, which describes the linking of the independent measurands x_i, is a sum. The partial derivatives are all equal to one and the following is produced:

   $$s_z^2 = s_1^2 + s_2^2 + \ldots + s_k^2 .$$

   The following is correspondingly applicable to the variances of the accompanying populations:

   $$\sigma_z^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_k^2 .$$

   The variance of an indirect measurand which is determined by the addition of independent individual measurands is equal to the sum of the variances of the individual measurands. This fact is made use of within the scope of statistical toleranc-ing. (compare with [4]).

3. A simple application case similar to no. 2 can be found in association with the calculation of control limits within the scope of Statistical Process Control [5].

   For $z = \bar{x} = \frac{1}{n} \cdot (x_1 + x_2 + \ldots + x_n)$ all partial derivatives are $\frac{\partial z}{\partial x_i} = \frac{1}{n}$.

   If $\sigma_1^2 = \sigma_2^2 = \ldots = \sigma_k^2$, then the following ensues: $\sigma_{\bar{x}} = \frac{\sigma_x}{\sqrt{n}}$ (compare with Sec. 5.).

   This means that the standard deviation of the average of n individual values is $\frac{1}{\sqrt{n}}$ smaller than the standard deviation of the individual values.

11 Sampling Scheme for Continuous Characteristics

A sampling scheme is a combination of short instructions which produce a decision about the acceptance or rejection of a lot.

In the application of the Bosch single-sampling scheme (compare with [2]) for the inspection of discrete characteristics (e.g. within the scope of receiving inspection), delivered lots are accepted when no nonconforming part is found within the sample (acceptance number c = 0).

This type of inspection enables no statement on the distribution of characteristics of the delivered lot and its statistical characteristics.

In the case of quantitative (measurable) characteristics the application of the scheme described below provides an alternative to the inspection of discrete characteristics according to the principle “To measure what is measurable”. A prerequisite for its application is, therefore, that the interesting characteristic is a measurable quantity.

For inspection of continuous characteristics, indicating measuring instruments are needed which are suitable for the inspection (capable).

Examples for the application:

- destructive or very expensive inspection
- Quality inspections (tests)
- receiving inspections

The advantages are:

- Possibility for assessment of the distribution of the characteristic and of the location of the individual values within the tolerance zone.
- For identical operating characteristics, the sample size is smaller than in the case of inspection by attributes.

Prerequisite: The considered quality characteristic is approximately normally distributed.

BOSCH

Sampling Scheme for Inspection of Continuous Characteristics

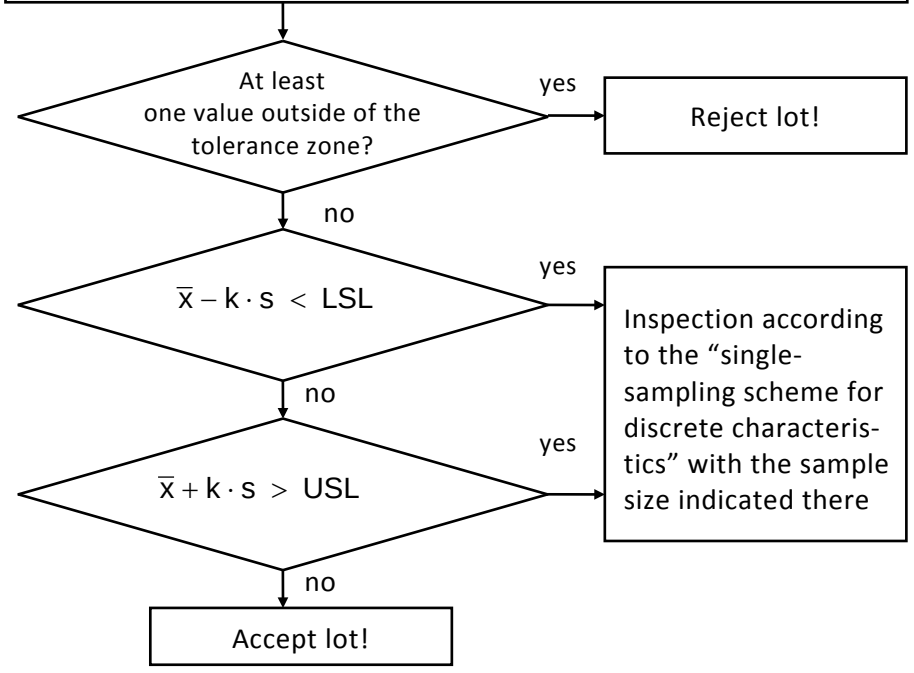
It is not permissible to use the sample sizes indicated here for inspection by attributes.

Hint: If there are justified doubts with respect to normality of the measured values the “single-sampling scheme for discrete characteristics” or 100% inspection is to be applied.

Lot size N		101..... 250	251..... 500	501..... 1000	1001..... 2000	2001..... ..≥ 5000
corresp.	n	100 %	39	42	44	46
to	k	-	3.0	3.0	3.1	3.1
Test EX	p'_{90}	-	0.03	0.02	0.02	0.018
discrete	p'_{10}	-	0.7	0.5	0.5	0.39
corresp.	n	27	27	30	33	35
to	k	2.6	2.6	2.7	2.8	2.9
Test EV	p'_{90}	0.08	0.08	0.06	0.05	0.04
discrete	p'_{10}	1.8	1.7	1.3	1.1	0.9
corresp.	n	19	21	23	25	27
to	k	2.4	2.4	2.5	2.6	2.6
Test EII	p'_{90}	0.17	0.14	0.11	0.1	0.08
discrete	p'_{10}	3.6	3.0	2.4	2.0	1.7
corresp.	n	12	13	16	17	19
to	k	2.0	2.1	2.2	2.3	2.4
Test EI	p'_{90}	0.42	0.34	0.26	0.21	0.17
discrete	p'_{10}	8.7	7.2	5.5	4.4	3.7

n	Sample size
k	Acceptability constant
p'_{90}	Nonconforming proportion in the lot (in %) for which the lot has 90 % acceptance probability
p'_{10}	Nonconforming proportion in the lot (in %) for which the lot has 10 % acceptance probability

1. Find sample size n and acceptability constant k for the given lot size N and the desired test in the table.
2. Draw a sample of n units from the considered lot!
3. Measure the interesting characteristic of these units!
4. Record the measurement values!
5. Calculate the mean $\bar{x}$ and the standard deviation s of these values!



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Example of procedure (cf. flow chart):

For a flange diameter the target value $C = 22.5\,mm$, the lower specification limit $LSL = 22.35\,mm$ and the upper specification limit $USL = 22.65\,mm$ are given.

1. *An inspection according to the test EV should decide on acceptance or rejection of a lot of $N = 480$ parts.*

   In the table one finds the values $n = 27$ and $k = 2.6$ in the row "Test EV" and the column corresponding to the lot size $N = 251.....500$.

2. *A sample of $n = 27$ parts is drawn from the lot.*

3. *The interesting characteristic of the parts is measured.*

4. *The measurements have the following results:*

Flange diameter in mm									
22.57	*22.52*	*22.54*	*22.51*	*22.56*	*22.57*	*22.55*	*22.61*	*22.54*	*22.53*
22.55	*22.56*	*22.54*	*22.57*	*22.56*	*22.58*	*22.54*	*22.56*	*22.53*	*22.57*
22.55	*22.56*	*22.57*	*22.54*	*22.55*	*22.56*	*22.56*			

   A probability plot for these results shows that the values are approximately normally distributed.

5. *The calculation of the mean and the standard deviation produces:*

   $\bar{x} = 22.55\ mm \qquad s = 0.02\,mm$

No value is out of tolerance.

$\bar{x} - k \cdot s = 22.55 - 2.6 \cdot 0.02 = 22.5 \geq 22.35 = LSL$

$\bar{x} + k \cdot s = 22.55 + 2.6 \cdot 0.03 = 22.6 \leq 22.65 = USL$

Decision: The lot is accepted.

HINTS:

1. *The values for k and n indicated in the table are calculated in such a way that the same operation characteristic is produced as is valid for the corresponding inspection of a discrete characteristic (equal values p'_{90} and p'_{10} respectively). For the selection of a test (individual inspection instruction) the applicant can orientate himself to the quantity p'_{90}. If this is the nonconforming fraction in the lot, the lot is accepted with 90 % probability. The smaller p'_{90} the more tight is the inspection.*

2. *Normality of the quality characteristic is a prerequisite for the application of inspection using continuous characteristics. As a result of non-compliance of this prerequisite the case can occur where $\bar{x} - k \cdot s < LSL$ and/or $\bar{x} + k \cdot s > USL$ despite the fact that the characteristic values of all parts of the lot are within the tolerance zone. This is possible for example if the lot has been inspected as to both sides by the supplier.*

   In order not to unjustly reject a totally conforming lot in this case, the single sampling scheme for inspection of discrete characteristics with the sample size indicated there must be applied in the interests of certainty.

3. *The previous text refers only to one inspection characteristic. The literature in this section also deals with only one measurement quantity. In the case of several characteristics to be inspected, each of these characteristics must be measured on each part of the sample. One then obtains a measurement series per characteristic. The steps given from no. 3 on are to be implemented for each characteristic.*

4. According to [27], the procedure is only permissible if p'_{90} concerns the exceeding of a one-sided limit. As long as p'_{90} concerns the exceeding of a two-sided limit in the case of two-sided limited characteristics, a graphic procedure given in [27] for "combined double specification limits" must be used (compare with the hint in [25] Sections 1.2.2 and 2.). This difference is, however, negligible for $p'_{90} < 1\%$.

Since the average and the standard deviation of the inspected population are unknown, but are rather estimated via $\bar{x}$ and s of the sample, the mathematical theory for the sampling scheme for inspection by continuous characteristics is based on the non-central t distribution. A description of the mathematical background can be found in [22], for example.

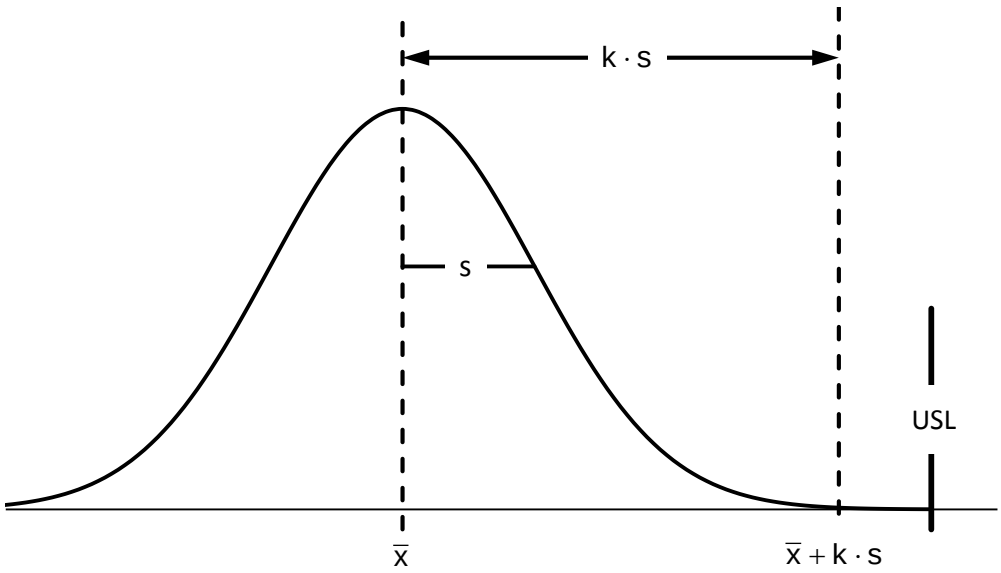

Fig. 11.1: For the illustration of the rejection criterion $\bar{x} + k \cdot s > USL$

12 Tables

12.1 Standard normal distribution Φ(u) (one-sided left)

u	0	1	2	3	4	5	6	7	8	9
	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
- 0.0	5000	4960	4920	4880	4840	4801	4761	4721	4681	4641
- 0.1	4602	4562	4522	4483	4443	4404	4364	4325	4286	4247
- 0.2	4207	4168	4129	4090	4052	4013	3974	3936	3897	3859
- 0.3	3821	3783	3745	3707	3669	3632	3594	3557	3520	3483
- 0.4	3446	3409	3372	3336	3300	3264	3228	3192	3156	3121
- 0.5	3085	3050	3015	2981	2946	2912	2877	2843	2810	2776
- 0.6	2743	2709	2676	2643	2611	2578	2546	2514	2483	2451
- 0.7	2420	2389	2358	2327	2297	2266	2236	2206	2177	2148
- 0.8	2119	2090	2061	2033	2005	1977	1949	1922	1894	1867
- 0.9	1841	1814	1788	1762	1736	1711	1685	1660	1635	1611
- 1.0	1587	1562	1539	1515	1492	1469	1446	1423	1401	1379
- 1.1	1357	1335	1314	1292	1271	1251	1230	1210	1190	1170
- 1.2	1151	1131	1112	1093	1075	1056	1038	1020	1003	0985
- 1.3	0968	0951	0934	0918	0901	0885	0869	0853	0838	0823
- 1.4	0808	0793	0778	0764	0749	0735	0721	0708	0694	0681
- 1.5	0668	0655	0643	0630	0618	0606	0594	0582	0571	0559
- 1.6	0548	0537	0526	0516	0505	0495	0485	0475	0465	0455
- 1.7	0446	0436	0427	0418	0409	0401	0392	0384	0375	0367
- 1.8	0359	0351	0344	0336	0329	0322	0314	0307	0301	0294
- 1.9	0287	0281	0274	0268	0262	0256	0250	0244	0239	0233
- 2.0	0228	0222	0217	0212	0207	0202	0197	0192	0188	0183
- 2.1	0179	0174	0170	0166	0162	0158	0154	0150	0146	0143
- 2.2	0139	0136	0132	0129	0125	0122	0119	0116	0113	0110
- 2.3	0107	0104	0102	0099	0096	0094	0091	0089	0087	0084
- 2.4	0087	0080	0078	0075	0073	0071	0069	0068	0066	0064
- 2.5	0062	0060	0059	0057	0055	0054	0052	0051	0049	0048
- 2.6	0047	0045	0044	0043	0041	0040	0039	0038	0037	0036
- 2.7	0035	0034	0033	0032	0031	0030	0029	0028	0027	0026
- 2.8	0026	0025	0024	0023	0023	0022	0021	0021	0020	0019
- 2.9	0019	0018	0018	0017	0016	0016	0015	0015	0014	0014
- 3.0	0013	0013	0013	0012	0012	0011	0011	0011	0010	0010
- 3.1	0010	0009	0009	0009	0008	0008	0008	0008	0007	0007
- 3.2	0007	0007	0006	0006	0006	0006	0006	0005	0005	0005
- 3.3	0005	0005	0005	0004	0004	0004	0004	0004	0004	0003
- 3.4	0003	0003	0003	0003	0003	0003	0003	0003	0003	0002

u corresponds to test statistic z

Standard normal distribution $\Phi(u)$ (one-sided right)

u	0	1	2	3	4	5	6	7	8	9
	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.0	5000	5040	5080	5120	5160	5199	5239	5279	5319	5359
0.1	5398	5438	5478	5517	5557	5596	5636	5675	5714	5733
0.2	5793	5832	5871	5910	5948	5987	6026	6064	6103	6141
0.3	6179	6217	6255	6293	6331	6368	6406	6443	6480	6517
0.4	6554	6591	6628	6664	6700	6736	6772	6808	6844	6879
0.5	6915	6950	6985	7019	7054	7088	7123	7157	7190	7224
0.6	7257	7291	7324	7357	7389	7422	7454	7486	7517	7549
0.7	7580	7611	7642	7673	7703	7734	7764	7794	7823	7852
0.8	7881	7910	7939	7967	7995	8023	8051	8078	8106	8133
0.9	8159	8186	8212	8238	8264	8289	8315	8340	8365	8389
1.0	8413	8438	8461	8485	8508	8531	8554	8577	8599	8621
1.1	8643	8665	8686	8708	8729	8749	8770	8790	8810	8830
1.2	8849	8869	8888	8907	8925	8944	8962	8980	8997	9015
1.3	9032	9049	9066	9082	9099	9115	9131	9147	9162	9177
1.4	9192	9207	9222	9236	9251	9265	9279	9292	9306	9319
1.5	9332	9345	9357	9370	9382	9394	9406	9418	9429	9441
1.6	9452	9463	9474	9484	9495	9505	9515	9525	9535	9545
1.7	9554	9564	9573	9582	9591	9599	9608	9616	9625	9633
1.8	9641	9649	9656	9664	9671	9678	9686	9693	9699	9706
1.9	9713	9719	9726	9732	9738	9744	9750	9756	9761	9767
2.0	9772	9778	9783	9788	9793	9798	9803	9808	9812	9817
2.1	9821	9826	9830	9834	9838	9842	9846	9850	9854	9857
2.2	9861	9864	9868	9871	9875	9878	9881	9884	9887	9890
2.3	9893	9896	9898	9901	9904	9906	9909	9911	9913	9916
2.4	9918	9920	9922	9925	9927	9929	9931	9932	9934	9936
2.5	9938	9940	9941	9943	9945	9946	9948	9949	9951	9952
2.6	9953	9955	9956	9957	9959	9960	9961	9962	9963	9964
2.7	9965	9966	9967	9968	9969	9970	9971	9972	9973	9974
2.8	9974	9975	9976	9977	9977	9978	9979	9979	9980	9981
2.9	9981	9982	9982	9983	9984	9984	9985	9985	9986	9986
3.0	9987	9987	9987	9988	9988	9989	9989	9989	9990	9990
3.1	9990	9991	9991	9991	9992	9992	9992	9992	9993	9993
3.2	9993	9993	9994	9994	9994	9994	9994	9995	9995	9995
3.3	9995	9995	9996	9996	9996	9996	9996	9996	9996	9997
3.4	9997	9997	9997	9997	9997	9997	9997	9997	9997	9998

u corresponds to test statistic z

Standard normal distribution $\Phi(u)$ (two-sided)

u	0	1	2	3	4	5	6	7	8	9
	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
0.0	0000	0080	0160	0239	0319	0399	0478	0558	0638	0717
0.1	0797	0876	0955	1034	1113	1192	1271	1350	1429	1507
0.2	1585	1663	1741	1819	1897	1974	2051	2128	2205	2282
0.3	2358	2434	2510	2586	2661	2737	2812	2886	2961	3035
0.4	3108	3182	3255	3328	3401	3473	3545	3616	3688	3759
0.5	3829	3900	3969	4039	4108	4177	4245	4313	4381	4448
0.6	4515	4581	4647	4713	4778	4843	4908	4971	5035	5098
0.7	5161	5223	5285	5346	5407	5468	5528	5587	5646	5705
0.8	5763	5821	5878	5935	5991	6047	6102	6157	6211	6265
0.9	6319	6372	6424	6476	6528	6579	6629	6680	6729	6778
1.0	6827	6875	6923	6969	7017	7063	7109	7154	7199	7243
1.1	7287	7330	7373	7415	7457	7499	7540	7580	7620	7660
1.2	7699	7737	7775	7813	7850	7887	7923	7959	7995	8030
1.3	8064	8098	8132	8165	8198	8230	8262	8293	8324	8355
1.4	8385	8415	8444	8473	8501	8529	8557	8584	8611	8638
1.5	8664	8690	8715	8740	8764	8789	8812	8836	8859	8882
1.6	8904	8926	8948	8969	8990	9011	9031	9051	9070	9090
1.7	9109	9127	9146	9164	9181	9199	9216	9233	9249	9266
1.8	9281	9297	9312	9328	9342	9357	9371	9385	9399	9412
1.9	9426	9439	9451	9464	9476	9488	9500	9512	9523	9534
2.0	9545	9556	9566	9576	9586	9596	9606	9616	9625	9634
2.1	9643	9651	9660	9668	9677	9684	9692	9700	9707	9715
2.2	9722	9729	9736	9743	9749	9756	9762	9768	9774	9780
2.3	9786	9791	9797	9802	9807	9812	9817	9822	9827	9832
2.4	9836	9841	9845	9849	9853	9857	9861	9865	9869	9872
2.5	9876	9879	9883	9886	9889	9892	9895	9898	9901	9904
2.6	9907	9910	9912	9915	9917	9920	9922	9924	9926	9929
2.7	9931	9933	9935	9937	9939	9940	9942	9944	9946	9947
2.8	9949	9951	9952	9954	9955	9956	9958	9959	9960	9962
2.9	9963	9964	9965	9966	9967	9968	9969	9970	9971	9972
3.0	9973	9974	9975	9976	0076	9977	9978	9979	9979	9980
3.1	9981	9981	9982	9983	9983	9984	9984	9984	9985	9986
3.2	9986	9987	9987	9988	9988	9989	9989	9989	9990	9990
3.3	9990	9991	9991	9991	9992	9992	9992	9992	9993	9993

u corresponds to test statistic z

12.2 t distribution (one-sided)

	Significance level			
Degrees of freedom f	90 %	95 %	99 %	99.9 %
1	3.078	6.314	31.82	318.30
2	1.886	2.920	6.965	22.33
3	1.638	2.353	4.541	10.21
4	1.533	2.132	3.747	7.173
5	1.476	2.015	3.365	5.893
6	1.440	1.943	3.143	5.208
7	1.415	1.895	2.998	4.785
8	1.397	1.860	2.896	4.501
9	1.383	1.833	2.821	4.297
10	1.372	1.812	2.764	4.144
11	1.363	1.796	2.718	4.025
12	1.356	1.782	2.681	3.930
13	1.350	1.771	2.650	3.852
14	1.345	1.761	2.624	3.787
15	1.341	1.753	2.602	3.733
16	1.337	1.746	2.583	3.686
17	1.333	1.740	2.567	3.646
18	1.330	1.734	2.552	3.610
19	1.328	1.729	2.539	3.579
20	1.325	1.725	2.528	3.552
21	1.323	1.721	2.518	3.527
22	1.321	1.717	2.508	3.505
23	1.319	1.714	2.500	3.485
24	1.318	1.711	2.492	3.467
25	1.316	1.708	2.485	3.450
26	1.315	1.706	2.479	3.435
27	1.314	1.703	2.473	3.421
28	1.313	1.701	2.467	3.408
29	1.311	1.699	2.462	3.396
30	1.310	1.697	2.457	3.385
40	1.303	1.684	2.423	3.307
50	1.299	1.676	2.403	3.261
60	1.296	1.671	2.390	3.232
80	1.292	1.664	2.374	3.195
100	1.290	1.660	2.364	3.174
200	1.286	1.652	2.345	3.131
500	1.283	1.648	2.334	3.107
∞	1.282	1.645	2.326	3.090

t distribution (two-sided)

	Significance level					
f	80 %	90 %	95 %	99 %	99.8 %	99.9 %
1	3.078	6.314	12.71	63.66	318.30	636.600
2	1.886	2.920	4.303	9.925	22.33	31.60
3	1.638	2.353	3.182	5.841	10.21	12.92
4	1.533	2.132	2.776	4.604	7.173	8.610
5	1.476	2.015	2.571	4.032	5.893	6.869
6	1.440	1.943	2.447	3.707	5.208	5.959
7	1.415	1.895	2.365	3.499	4.785	5.408
8	1.397	1.860	2.306	3.355	4.501	5.041
9	1.383	1.833	2.262	3.250	4.297	4.781
10	1.372	1.812	2.228	3.169	4.144	4.587
11	1.363	1.796	2.201	3.106	4.025	4.437
12	1.356	1.782	2.179	3.055	3.930	4.318
13	1.350	1.771	2.160	3.012	3.852	4.221
14	1.345	1.761	2.145	2.977	3.787	4.140
15	1.341	1.753	2.131	2.947	3.733	4.073
16	1.337	1.746	2.120	2.921	3.686	4.015
17	1.333	1.740	2.110	2.898	3.646	3.965
18	1.330	1.734	2.101	2.878	3.610	3.922
19	1.328	1.729	2.093	2.861	3.579	3.883
20	1.325	1.725	2.086	2.845	3.552	3.850
21	1.323	1.721	2.080	2.831	3.527	3.819
22	1.321	1.717	2.074	2.819	3.505	3.792
23	1.319	1.714	2.069	2.807	3.485	3.768
24	1.318	1.711	2.064	2.797	3.467	3.745
25	1.316	1.708	2.060	2.787	3.450	3.725
26	1.315	1.706	2.056	2.779	3.435	3.707
27	1.314	1.703	2.052	2.771	3.421	3.690
28	1.313	1.701	2.048	2.763	3.408	3.674
29	1.311	1.699	2.045	2.756	3.396	3.659
30	1.310	1.697	2.042	2.750	3.385	3.646
40	1.303	1.684	2.021	2.704	3.307	3.551
50	1.299	1.676	2.009	2.678	3.261	3.496
60	1.296	1.671	2.000	2.660	3.232	3.460
80	1.292	1.664	1.990	2.639	3.195	3.416
100	1.290	1.660	1.984	2.626	3.174	3.390
200	1.286	1.652	1.972	2.601	3.131	3.340
500	1.283	1.648	1.965	2.586	3.107	3.310
∞	1.282	1.645	1.960	2.576	3.090	3.291

12.3 F distribution (Significance level 90 %, one-sided)

	f_1											
f_2	1	2	3	4	5	6	8	10	12	20	50	∞
1	39.9	49.5	53.6	55.8	57.2	58.2	59.4	60.2	60.7	61.7	62.7	63.3
2	8.53	9.00	9.16	9.24	9.29	9.33	9.37	9.39	9.41	9.44	9.47	9.49
3	5.54	5.46	5.39	5.34	5.31	5.28	5.25	5.23	5.22	5.18	5.15	5.13
4	4.54	4.32	4.19	4.11	4.05	4.01	3.95	3.92	3.90	3.84	3.80	3.76
5	4.06	3.78	3.62	3.52	3.45	3.40	3.34	3.30	3.27	3.21	3.15	3.10
6	3.78	3.46	3.29	3.18	3.11	3.05	2.98	2.94	2.90	2.84	2.77	2.72
7	3.59	3.26	3.07	2.96	2.88	2.83	2.75	2.70	2.67	2.59	2.52	2.47
8	3.46	3.11	2.92	2.81	2.73	2.67	2.59	2.54	2.50	2.42	2.35	2.29
9	3.36	3.01	2.81	2.69	2.61	2.56	2.47	2.42	2.38	2.30	2.22	2.16
10	3.29	2.92	2.73	2.61	2.52	2.46	2.38	2.32	2.28	2.20	2.12	2.06
12	3.18	2.81	2.61	2.48	2.39	2.33	2.24	2.19	2.15	2.06	1.97	1.90
14	3.10	2.73	2.52	2.39	2.31	2.24	2.15	2.10	2.05	2.96	1.87	1.80
16	3.05	2.67	2.46	2.33	2.24	2.18	2.09	2.03	1.99	1.89	1.79	1.72
18	3.01	2.62	2.42	2.29	2.20	2.13	2.04	1.95	1.93	1.84	1.74	1.66
20	2.97	2.59	2.38	2.25	2.16	2.09	2.00	1.94	1.89	1.79	1.69	1.61
25	2.92	2.53	2.32	2.18	2.09	2.02	1.93	1.87	1.82	1.72	1.61	1.52
30	2.88	2.49	2.28	2.14	2.05	1.98	1.88	1.82	1.77	1.67	1.55	1.46
35	2.85	2.46	2.25	2.11	2.02	1.95	1.85	1.79	1.74	1.63	1.51	1.41
40	2.84	2.44	2.23	2.09	2.00	1.93	1.83	1.76	1.71	1.61	1.48	1.38
45	2.82	2.42	2.21	2.07	1.98	1.91	1.81	1.74	1.70	1.58	1.46	1.35
50	2.81	2.41	2.20	2.06	1.97	1.90	1.80	1.73	1.68	1.57	1.44	1.33
60	2.79	2.39	2.18	2.04	1.95	1.87	1.77	1.71	1.66	1.54	1.41	1.29
80	2.77	2.37	2.15	2.02	1.92	1.85	1.75	1.68	1.63	1.51	1.38	1.24
100	2.76	2.36	2.14	2.00	1.91	1.83	1.73	1.66	1.61	1.49	1.35	1.21
∞	2.71	2.30	2.08	1.94	1.85	1.77	1.67	1.60	1.55	1.42	1.26	1.00

F distribution (P_A = 95 %, one-sided)

	f_1											
f_2	1	2	3	4	5	6	8	10	12	20	50	∞
1	161	200	216	225	230	234	239	242	244	248	252	254
2	18.5	19.0	19.2	19.2	19.3	19.3	19.4	19.4	19.4	19.4	19.5	19.5
3	10.1	9.6	9.3	9.1	9.0	8.9	8.9	8.8	8.7	8.7	8.6	8.53
4	7.71	6.94	6.59	6.39	6.26	6.16	6.04	5.96	5.91	5.80	5.70	5.63
5	6.61	5.79	5.41	5.19	5.05	4.95	4.82	4.74	4.68	4.56	4.44	4.36
6	5.99	5.14	4.76	4.53	4.39	4.28	4.15	4.06	4.00	3.87	3.75	3.67
7	5.59	4.74	4.35	4.12	3.97	3.87	3.73	3.64	3.57	3.44	3.32	3.23
8	5.32	4.46	4.07	3.84	3.69	3.58	3.44	3.35	3.28	3.15	3.02	2.93
9	5.12	4.26	3.86	3.63	3.48	3.37	3.23	3.14	3.07	2.94	2.80	2.71
10	4.96	4.10	3.71	3.48	3.33	3.22	3.07	2.98	2.91	2.77	2.64	2.54
12	4.75	3.89	3.49	3.26	3.11	3.00	2.85	2.75	2.69	2.54	2.40	2.30
14	4.60	3.74	3.34	3.11	2.96	2.85	2.70	2.60	2.53	2.39	2.24	2.13
16	4.49	3.63	3.24	3.01	2.85	2.74	2.59	2.49	2.42	2.28	2.12	2.01
18	4.41	3.55	3.16	2.93	2.77	2.66	2.51	2.41	2.34	2.19	2.04	1.92
20	4.35	3.49	3.10	2.87	2.71	2.60	2.45	2.35	2.28	2.12	1.97	1.84
25	4.24	3.39	2.99	2.76	2.60	2.49	2.34	2.24	2.16	2.01	1.84	1.71
30	4.17	3.32	2.92	2.69	2.53	2.42	2.27	2.16	2.09	1.93	1.76	1.62
35	4.12	3.27	2.87	2.64	3.49	2.37	2.22	2.11	2.04	1.88	1.70	1.56
40	4.08	3.23	2.84	2.61	2.45	2.34	2.18	2.08	2.00	1.84	1.66	1.51
45	4.06	3.20	2.81	2.58	2.42	2.31	2.15	2.05	1.97	1.81	1.63	1.47
50	4.03	3.18	2.79	2.56	2.40	2.29	2.13	2.03	1.95	1.78	1.60	1.44
60	4.00	3.15	2.76	2.53	2.37	2.25	2.10	1.99	1.92	1.75	1.56	1.39
70	3.98	3.13	2.74	2.50	2.35	2.23	2.07	1.97	1.89	1.72	1.53	
80	3.96	3.11	2.72	2.49	2.33	2.21	2.06	1.95	1.88	1.70	1.51	1.32
90	3.95	3.10	2.71	2.47	2.32	2.20	2.04	1.94	1.86	1.69	1.49	
100	3.94	3.09	2.70	2.46	2.31	2.19	2.03	1.93	1.85	1.68	1.48	1.28
150	3.90	3.06	2.66	2.43	2.27	2.16	2.00	1.89	1.82	1.64	1.44	
200	3.89	3.04	2.65	2.42	2.26	2.14	1.98	1.88	1.80	1.62	1.41	
300	3.87	3.03	2.63	2.40	2.24	2.13	1.97	1.86	1.78	1.61	1.39	
500	3.86	3.01	2.62	2.39	2.23	2.12	1.96	1.85	1.77	1.59	1.38	
∞	3.84	3.00	2.60	2.37	2.21	2.10	1.94	1.83	1.75	1.57	1.35	1.00

F distribution (P_A = 99 %, one-sided)

	f_1											
f_2	1	2	3	4	5	6	8	10	12	20	50	∞
2	98.5	99.0	99.2	99.2	99.3	99.3	99.4	99.4	99.4	99.4	99.5	99.5
3	34.1	30.8	29.5	29.5	28.2	27.9	27.5	27.2	27.1	26.7	26.4	26.1
4	21.2	18.0	16.7	16.0	15.5	15.2	14.8	14.5	14.4	14.0	13.7	13.5
5	16.3	13.3	12.1	11.4	11.0	10.7	10.3	10.1	9.89	9.55	9.24	9.02
6	13.8	10.9	9.78	9.15	8.75	8.47	8.10	7.87	7.72	7.40	7.09	6.88
7	12.3	9.55	8.45	7.85	7.46	7.19	6.84	6.62	6.47	6.16	5.86	5.65
8	11.3	8.65	7.59	7.01	6.63	6.37	6.03	5.81	5.67	5.36	5.07	4.86
9	10.6	8.02	6.99	6.42	6.06	5.80	5.47	5.26	5.11	4.81	4.52	4.31
10	10.0	7.56	6.55	5.99	5.64	5.39	5.06	4.85	4.71	4.41	4.12	3.91
12	9.33	6.93	5.95	5.41	5.06	4.82	4.50	4.30	4.16	3.86	3.57	3.36
14	8.86	6.51	5.56	5.04	4.69	4.46	4.14	3.94	3.80	3.51	3.22	3.00
16	8.53	6.23	5.29	4.77	4.44	4.20	3.89	3.69	3.55	3.26	2.97	2.75
18	8.29	6.01	5.09	4.58	4.25	4.01	3.71	3.51	3.37	3.08	2.78	2.57
20	8.10	5.85	4.94	4.43	4.10	3.87	3.56	3.37	3.23	2.94	2.64	2.42
25	7.77	5.57	4.68	4.18	3.86	3.63	3.32	3.13	2.99	2.70	2.40	2.17
30	7.56	5.39	4.51	4.02	3.70	3.47	3.17	2.98	2.84	2.55	2.25	2.01
35	7.42	5.27	4.40	3.91	3.59	3.37	3.07	2.88	2.74	2.44	2.14	1.89
40	7.31	5.18	4.31	3.83	3.51	3.29	3.99	2.80	2.66	2.37	2.06	1.80
45	7.23	5.11	4.25	3.77	3.45	3.23	2.94	2.74	2.61	2.31	2.00	1.74
50	7.17	5.06	4.20	3.72	3.41	3.19	2.89	2.70	2.56	2.27	1.95	1.68
60	7.08	4.98	4.13	3.65	3.34	3.12	2.82	2.63	2.50	2.20	1.88	1.60
70	7.01	4.92	4.08	3.60	3.29	3.07	2.78	2.59	2.45	2.15	1.83	
80	6.96	4.88	4.04	3.56	3.26	3.04	2.74	2.55	2.42	2.12	1.79	1.49
90	6.93	4.85	4.01	3.54	2.23	3.01	2.72	2.52	2.39	2.09	1.76	
100	6.90	4.82	3.98	3.51	3.21	2.99	2.69	2.50	2.37	2.07	1.73	1.43
150	6.81	4.75	3.92	3.45	3.14	2.92	2.63	2.44	2.31	2.00	1.66	
200	6.76	4.71	3.88	3.41	3.11	2.89	2.60	2.41	2.27	1.97	1.63	
300	6.72	4.68	3.85	3.38	3.08	2.86	2.57	2.38	2.24	1.94	1.59	
500	6.69	4.65	3.82	3.36	3.05	2.84	2.55	2.36	2.22	1.92	1.56	
∞	6.63	4.61	3.78	3.32	3.02	2.80	2.51	2.32	2.18	1.88	1.52	1.00

F distribution (P_A = 95 %, two-sided)

	f_1								
f_2	1	2	3	4	5	6	7	8	9
1	648	800	864	900	922	937	948	957	963
2	38.5	39.0	39.2	39.2	39.3	39.3	39.4	39.4	39.4
3	17.4	16.0	15.4	15.1	14.9	14.7	14.6	14.5	14.5
4	12.2	10.6	9.98	9.60	9.36	9.20	9.07	8.98	8.90
5	10.0	8.43	7.76	7.39	7.15	6.98	6.85	6.76	6.68
6	8.81	7.26	6.60	6.23	5.99	5.82	5.70	5.60	5.52
7	8.07	6.54	5.89	5.52	5.29	5.12	4.99	4.90	4.82
8	7.57	6.06	5.42	5.05	4.82	4.65	4.53	4.43	4.36
9	7.21	5.71	5.08	4.72	4.48	4.32	4.20	4.10	4.03
10	6.94	5.46	4.83	4.47	4.24	4.07	3.95	3.85	3.78
11	6.72	5.25	4.63	4.27	4.04	3.88	3.76	3.66	3.59
12	6.55	5.10	4.47	4.12	3.89	3.73	3.61	3.51	3.44
13	6.41	4.97	4.35	4.00	3.77	3.60	3.48	3.39	3.31
14	6.30	4.86	4.24	3.89	3.66	3.50	3.38	3.29	3.21
15	6.20	4.77	4.15	3.80	3.58	3.41	3.29	3.20	3.12
16	6.12	4.69	4.08	3.73	3.50	3.34	3.22	3.12	3.05
17	6.04	4.62	4.01	3.66	3.44	3.28	3.16	3.06	2.98
18	5.98	4.56	3.95	3.61	3.38	3.22	3.10	3.01	2.93
19	5.92	4.51	3.90	3.56	3.33	3.17	3.05	2.96	2.88
20	5.87	4.46	3.86	3.51	3.29	3.13	3.01	2.91	2.84
22	5.79	4.38	3.78	3.44	3.22	3.05	2.93	2.84	2.76
24	5.72	4.32	3.72	3.38	3.15	2.99	2.87	2.78	2.70
26	5.66	4.27	3.67	3.33	3.10	2.94	2.82	2.73	2.65
28	5.61	4.22	3.63	3.29	3.06	2.90	2.78	2.69	2.61
30	5.57	4.18	3.59	3.25	3.03	2.87	2.75	2.65	2.57
40	5.42	4.05	3.46	3.13	2.90	2.74	2.62	2.53	2.45
50	5.34	3.97	3.39	3.05	2.83	2.67	2.55	2.46	2.38
60	5.29	3.93	3.34	3.01	2.79	2.63	2.51	2.41	2.33
70	5.25	3.89	3.31	2.97	2.75	2.59	2.47	2.38	2.30
80	5.22	3.86	3.28	2.95	2.73	2.57	2.45	2.35	2.28
90	5.20	3.84	3.26	2.93	2.71	2.55	2.43	2.34	2.26
100	5.18	3.83	3.25	2.92	2.70	2.54	2.42	2.32	2.24
200	5.09	3.76	3.18	2.85	2.63	2.47	2.35	2.26	2.18
500	5.05	3.72	3.14	2.81	2.59	2.43	2.31	2.22	2.14

F distribution (P_A = 95 %, two-sided), cont.

	f_1							
f_2	10	15	20	30	40	50	100	∞
1	969	985	993	1.001	1.006	1.008	1.013	1.018
2	39.4	39.4	39.4	39.5	39.5	39.5	39.5	39.5
3	14.4	14.3	14.2	14.1	14.0	14.0	14.0	13.9
4	8.84	8.66	8.56	8.46	8.41	8.38	8.32	8.26
5	6.62	6.43	6.33	6.23	6.18	6.14	6.08	6.02
6	5.46	5.27	5.17	5.07	5.01	4.98	4.92	4.85
7	4.76	4.57	4.47	4.36	4.31	4.28	4.21	4.14
8	4.30	4.10	4.00	3.89	3.84	3.81	3.74	3.67
9	3.96	3.77	3.67	3.56	3.51	3.47	3.40	3.33
10	3.72	3.52	3.42	3.31	3.26	3.22	3.15	3.08
11	3.52	3.34	3.22	3.12	3.06	3.03	2.95	2.88
12	3.37	3.18	3.07	2.96	2.91	2.87	2.80	2.72
13	3.25	3.05	2.95	2.84	2.78	2.74	2.67	2.60
14	3.15	2.95	2.84	2.73	2.67	2.64	2.56	2.49
15	3.06	2.86	2.76	2.64	2.58	2.55	2.47	2.40
16	2.99	2.79	2.68	2.57	2.51	2.47	2.40	2.32
17	2.92	2.72	2.62	2.50	2.44	2.41	2.33	2.25
18	2.87	2.67	2.56	2.44	2.38	2.35	2.27	2.19
19	2.82	2.62	2.51	2.39	2.33	2.30	2.22	2.13
20	2.77	2.57	2.46	2.35	2.29	2.25	2.17	2.09
22	2.70	2.50	2.39	2.27	2.21	2.17	2.09	2.00
24	2.64	2.44	2.33	2.21	2.15	2.11	2.02	1.94
26	2.59	2.39	2.28	2.16	2.09	2.05	1.97	1.88
28	2.55	2.34	2.23	2.11	2.05	2.01	1.92	1.83
30	2.51	2.31	2.20	2.07	2.01	1.97	1.88	1.79
40	2.39	2.18	2.07	1.94	1.88	1.83	1.74	1.64
50	2.32	2.11	1.99	1.87	1.80	1.75	1.66	1.55
60	2.27	2.06	1.94	1.82	1.74	1.70	1.60	1.48
70	2.24	2.03	1.91	1.78	1.71	1.66	1.56	1.44
80	2.21	2.00	1.88	1.75	1.68	1.63	1.53	1.40
90	2.19	1.98	1.86	1.73	1.66	1.61	1.50	1.37
100	2.18	1.97	1.85	1.71	1.64	1.59	1.48	1.35
200	2.11	1.90	1.78	1.64	1.56	1.51	1.39	1.23
500	2.07	1.86	1.74	1.60	1.51	1.46	1.34	1.14

F distribution (P_A = 99 %, two-sided)

	f_1								
f_2	1	2	3	4	5	6	7	8	9
1	16200	20000	21600	22500	23100	23400	23700	23900	24100
2	198	199	199	199	199	199	199	199	199
3	55.6	49.8	47.4	46.2	45.3	44.8	44.4	44.1	43.8
4	31.3	26.3	24.3	23.2	22.5	22.0	21.6	21.4	21.1
5	22.8	18.3	16.5	15.6	14.9	14.5	14.2	14.0	13.8
6	18.6	14.5	12.9	12.0	11.5	11.1	10.8	10.6	10.4
7	16.2	12.4	10.9	10.1	9.52	9.16	8.89	8.68	8.51
8	14.7	11.0	9.60	8.80	8.30	7.95	7.69	7.50	7.34
9	13.6	10.1	8.72	7.96	7.47	7.13	6.89	6.69	6.54
10	12.8	9.43	8.08	7.34	6.87	6.54	6.30	6.12	5.97
11	12.2	8.91	7.60	6.88	6.42	6.10	5.86	5.68	5.54
12	11.8	8.51	7.23	6.52	6.07	5.76	5.52	5.35	5.20
13	11.4	8.19	6.93	6.23	5.79	5.48	5.25	5.08	4.93
14	11.1	7.92	6.68	6.00	5.56	5.26	5.03	4.86	4.72
15	10.8	7.70	6.48	5.80	5.37	5.07	4.85	4.67	4.54
16	10.6	7.51	6.30	5.64	5.21	4.91	4.69	4.52	4.38
17	10.4	7.35	6.16	5.50	5.07	4.78	4.56	4.39	4.25
18	10.2	7.21	6.03	5.37	4.96	4.66	4.44	4.28	4.14
19	10.1	7.09	5.92	5.27	4.85	4.56	4.34	4.18	4.04
20	9.94	6.99	5.82	5.17	4.76	4.47	4.26	4.09	3.96
22	9.73	6.81	5.65	5.02	4.61	4.32	4.11	3.94	3.81
24	9.55	6.66	5.52	4.89	4.49	4.20	3.99	3.83	3.69
26	9.41	6.54	5.41	4.79	4.38	4.10	3.89	3.73	3.60
28	9.28	6.44	5.32	4.70	4.30	4.02	3.81	3.65	3.52
30	9.18	6.35	5.24	4.62	4.23	3.95	3.74	3.58	3.45
40	8.83	6.07	4.98	4.37	3.99	3.71	3.51	3.35	3.22
50	8.63	5.90	4.83	4.23	3.85	3.58	3.38	3.22	3.09
60	8.49	5.80	4.73	4.14	3.76	3.49	3.29	3.13	3.01
70	8.40	5.72	4.66	4.08	3.70	3.43	3.23	3.08	2.95
80	8.33	5.67	4.61	4.03	3.65	3.39	3.19	3.03	2.91
90	8.28	5.62	4.57	3.99	3.62	3.35	3.15	3.00	2.87
100	8.24	5.59	4.54	3.96	3.59	3.33	3.13	2.97	2.85
200	8.06	5.44	4.40	3.84	3.47	3.21	3.01	2.86	2.73
500	7.95	5.36	4.33	3.76	3.40	3.14	2.94	2.79	2.66

F distribution (P_A = 99 %, two-sided), cont.

	f_1							
f_2	10	15	20	30	40	50	100	∞
1	24200	24600	24800	25000	25100	25200	25300	25500
2	199	199	199	199	199	199	199	200
3	43.7	43.1	42.8	42.5	42.4	42.2	42.0	41.8
4	21.0	20.4	20.2	19.9	19.8	19.7	19.5	19.3
5	13.6	13.1	12.9	12.7	12.5	12.5	12.3	12.1
6	10.3	9.81	9.59	9.36	9.24	9.17	9.03	8.88
7	8.38	7.97	7.75	7.53	7.42	7.35	7.22	7.08
8	7.21	6.81	6.61	6.40	6.29	6.22	6.09	5.95
9	6.42	6.03	5.83	5.62	5.52	5.45	5.32	5.19
10	5.85	5.47	5.27	5.07	4.97	4.90	4.77	4.64
11	5.42	5.05	4.86	4.65	4.55	4.49	4.36	4.23
12	5.09	4.72	4.53	4.33	4.23	4.17	4.04	3.90
13	4.82	4.46	4.27	4.07	3.97	3.91	3.78	3.65
14	4.60	4.25	4.06	3.86	3.76	3.70	3.57	3.44
15	4.42	4.07	3.88	3.69	3.58	3.52	3.39	3.26
16	4.27	3.92	3.73	3.54	3.44	3.37	3.25	3.11
17	4.14	3.79	3.61	3.41	3.31	3.25	3.12	2.98
18	4.03	3.68	3.50	3.30	3.20	3.14	3.01	2.87
19	3.93	3.59	3.40	3.21	3.11	3.04	2.91	2.78
20	3.85	3.50	3.32	3.12	3.02	2.96	2.83	2.69
22	3.70	3.36	3.18	2.98	2.88	2.82	2.69	2.55
24	3.59	3.25	3.06	2.87	2.77	2.70	2.57	2.43
26	3.49	3.15	2.97	2.77	2.67	2.61	2.47	2.33
28	3.41	3.07	2.89	2.69	2.59	2.53	2.39	2.25
30	3.34	3.01	2.82	2.63	2.52	2.46	2.32	2.18
40	3.12	2.78	2.60	2.40	2.30	2.23	2.09	1.93
50	2.99	2.65	2.47	2.27	2.16	2.10	1.95	1.79
60	2.90	2.57	2.39	2.19	2.08	2.01	1.86	1.69
70	2.85	2.51	2.33	2.13	2.02	1.95	1.80	1.62
80	2.80	2.47	2.29	2.08	1.97	1.90	1.75	1.56
90	2.77	2.44	2.25	2.05	1.94	1.87	1.71	1.52
100	2.74	2.41	2.23	2.02	1.91	1.84	1.68	1.49
200	2.63	2.30	2.11	1.91	1.79	1.71	1.54	1.31
500	2.56	2.23	2.04	1.84	1.72	1.64	1.46	1.18

12.4 Chi-squared function

	Degrees of freedom						
F(X)	1	2	3	4	5	6	7
0.001	0.00	0.00	0.02	0.09	0.21	0.38	0.60
0.005	0.00	0.01	0.07	0.21	0.41	0.68	0.99
0.010	0.00	0.02	0.11	0.30	0.55	0.87	1.24
0.025	0.00	0.05	0.22	0.48	0.83	1.24	1.69
0.050	0.00	0.10	0.35	0.71	1.15	1.64	2.17
0.100	0.02	0.21	0.58	1.06	1.61	2.20	2.83
0.250	0.10	0.58	1.21	1.92	2.67	3.45	4.25
0.500	0.45	1.39	2.37	3.36	4.35	5.35	6.35
0.750	1.32	2.77	4.11	5.39	6.63	7.84	9.04
0.900	2.71	4.61	6.25	7.78	9.24	10.64	12.02
0.950	3.84	5.99	7.81	9.49	11.07	12.59	14.07
0.975	5.02	7.38	9.35	11.14	12.83	14.45	16.01
0.990	6.63	9.21	11.34	13.28	15.00	16.81	18.48
0.995	7.88	10.60	12.84	14.86	16.75	18.55	20.28
0.999	10.83	13.82	16.27	18.47	20.52	22.46	24.32

	Degrees of freedom						
F(X)	8	9	10	11	12	13	14
0.001	0.86	1.15	1.48	1.83	2.21	2.62	3.04
0.005	1.34	1.73	2.16	2.60	3.07	3.57	4.07
0.010	1.65	2.00	2.56	3.05	3.57	4.11	4.66
0.025	2.18	2.70	3.25	3.82	4.40	5.01	5.63
0.050	2.73	3.33	3.94	4.57	5.23	5.89	6.57
0.100	3.49	4.17	4.87	5.58	6.30	7.04	7.79
0.250	5.07	5.90	6.74	7.58	8.44	9.30	10.17
0.500	7.34	8.34	9.34	10.34	11.34	12.34	13.34
0.750	10.22	11.39	12.55	13.70	14.85	15.98	17.12
0.900	13.36	14.08	15.99	17.28	18.55	19.81	21.06
0.950	15.51	16.92	18.31	19.68	21.03	22.36	23.68
0.975	17.53	19.02	20.48	21.92	23.34	24.74	26.12
0.990	20.09	21.67	23.21	24.73	26.22	27.69	29.14
0.995	21.96	23.59	25.19	26.76	28.30	29.82	31.32
0.999	26.13	27.88	29.59	31.26	32.91	34.52	36.12

Chi-squared function (cont.)

	Degrees of freedom						
F(X)	15	16	17	18	19	20	21
0.001	3.48	3.94	4.42	4.90	5.41	5.92	6.4
0.005	4.60	5.14	5.70	6.26	6.84	7.43	8.0
0.010	5.23	5.81	6.41	7.01	7.63	8.26	8.9
0.025	6.26	6.91	7.56	8.23	8.01	9.59	10.3
0.050	7.26	7.96	8.67	9.30	10.12	10.85	11.6
0.100	8.55	9.31	10.00	10.86	11.65	12.44	13.2
0.250	11.04	11.91	12.79	13.68	14.56	15.45	16.3
0.500	14.34	15.34	16.34	17.34	18.34	19.34	20.3
0.750	18.25	19.37	20.49	21.60	22.72	23.83	24.0
0.900	22.31	23.54	24.77	25.09	27.20	28.41	29.6
0.950	25.00	26.30	27.59	28.87	30.14	31.41	32.7
0.975	27.40	28.85	30.10	31.53	32.85	34.17	35.5
0.990	30.58	32.00	33.41	34.81	36.19	37.57	38.0
0.995	32.80	34.27	35.72	37.16	38.58	40.00	41.4
0.999	37.70	39.25	40.79	42.31	43.82	45.32	46.8

	Degrees of freedom						
F(X)	22	23	24	25	26	27	28
0.001	7.0	7.5	8.1	8.7	9.2	9.8	10.43
0.005	8.6	9.3	9.9	10.5	11.2	11.8	12.5
0.010	9.5	10.2	10.9	11.5	12.2	12.9	13.6
0.025	11.0	11.7	12.4	13.1	13.8	14.6	15.3
0.050	12.3	13.1	13.8	14.6	15.4	16.2	16.9
0.100	14.0	14.8	15.7	16.5	17.3	18.1	18.9
0.250	17.2	18.1	19.0	19.9	20.8	21.7	22.7
0.500	21.3	22.3	23.3	24.3	25.3	26.3	27.3
0.750	26.0	27.1	28.2	29.3	30.4	31.5	32.6
0.900	30.8	32.0	33.2	34.4	35.0	36.7	37.9
0.950	33.9	35.2	36.4	37.7	38.0	40.1	41.3
0.975	36.8	38.1	39.4	40.6	41.9	43.2	44.5
0.990	40.3	41.6	43.0	44.3	45.6	47.0	48.3
0.995	42.8	44.2	45.6	46.9	48.3	49.6	51.0
0.999	48.3	49.7	51.2	52.6	54.1	55.5	56.9

Chi-squared function (cont.)

	Degrees of freedom						
F(X)	29	30	40	50	60	70	80
0.001	11.0	11.6	17.9	24.7	31.7	39.0	46.5
0.005	13.1	13.8	20.7	28.0	35.5	43.3	51.2
0.010	14.3	15.00	22.2	29.7	37.5	45.4	53.5
0.025	16.0	16.8	24.4	32.4	40.5	48.8	57.2
0.050	17.7	18.5	26.5	34.8	43.2	51.7	60.4
0.100	19.8	20.6	29.1	37.7	46.5	55.3	64.3
0.250	23.6	24.5	33.7	42.9	52.3	61.7	71.1
0.500	28.3	29.3	39.3	49.3	59.3	69.3	79.3
0.750	33.7	34.8	45.6	56.3	67.0	77.6	88.1
0.900	39.1	40.3	51.8	63.2	74.4	85.5	96.6
0.950	42.6	43.8	55.8	67.5	79.1	90.5	101.9
0.975	45.7	47.0	59.3	71.4	83.3	95.0	106.6
0.990	49.6	50.9	63.7	76.2	88.4	100.4	112.3
0.995	52.3	53.7	66.8	79.5	92.0	104.2	116.3
0.999	58.3	59.7	73.4	86.7	99.6	112.3	124.8

12.5 Critical Values for the Outlier Test According to Grubbs with Normal Distribution

n	$P_A = 95\%$	$P_A = 99\%$	n	$P_A = 95\%$	$P_A = 99\%$	n	$P_A = 95\%$	$P_A = 99\%$
			41	2.877	3.251	81	3.134	3.525
			42	2.887	3.261	82	3.139	3.529
3	1.153	1.155	43	2.896	3.271	83	3.143	3.534
4	1.463	1.492	44	2.905	3.282	84	3.147	3.539
5	1.672	1.749	45	2.914	3.292	85	3.151	3.543
6	1.822	1.944	46	2.923	3.302	86	3.155	3.547
7	1.938	2.097	47	2.931	3.310	87	3.160	3.551
8	2.032	2.221	48	2.940	3.319	88	3.163	3.555
9	2.110	2.323	49	2.948	3.329	89	3.167	3.559
10	2.176	2.410	50	2.956	3.336	90	3.171	3.563
11	2.234	2.485	51	2.964	3.345	91	3.174	3.567
12	2.285	2.550	52	2.971	3.353	92	3.179	3.570
13	2.331	2.607	53	2.978	3.361	93	3.182	3.575
14	2.371	2.659	54	2.986	3.368	94	3.186	3.579
15	2.409	2.705	55	2.992	3.376	95	3.189	3.582
16	2.443	2.747	56	3.000	3.383	96	3.193	3.586
17	2.475	2.785	57	3.006	3.391	97	3.196	3.589
18	2.504	2.821	58	3.013	3.397	98	3.201	3.593
19	2.532	2.854	59	3.019	3.405	99	3.204	3.597
20	2.557	2.884	60	3.025	3.411	100	3.207	3.600
21	2.580	2.912	61	3.032	3.418	101	3.210	3.603
22	2.603	2.939	62	3.037	3.424	102	3.214	3.607
23	2.624	2.963	63	3.044	3.430	103	3.217	3.610
24	2.644	2.987	64	3.049	3.437	104	3.220	3.614
25	2.663	3.009	65	3.055	3.442	105	3.224	3.617
26	2.681	3.029	66	3.061	3.449	106	3.227	3.620
27	2.698	3.049	67	3.066	3.454	107	3.230	3.623
28	2.714	3.068	68	3.071	3.460	108	3.233	3.626
29	2.730	3.085	69	3.076	3.466	109	3.236	3.629
30	2.745	3.103	70	3.082	3.471	110	3.239	3.632
31	2.759	3.119	71	3.087	3.476	111	3.242	3.636
32	2.773	3.135	72	3.092	3.482	112	3.245	3.639
33	2.786	3.150	73	3.098	3.487	113	3.248	3.642
34	2.799	3.164	74	3.102	3.492	114	3.251	3.645
35	2.811	3.178	75	3.107	3.496	115	3.254	3.647
36	2.823	3.191	76	3.111	3.502	116	3.257	3.650
37	2.835	3.204	77	3.117	3.507	117	3.259	3.653
38	2.846	3.216	78	3.121	3.511	118	3.262	3.656
39	2.857	3.228	79	3.125	3.516	119	3.265	3.659
40	2.866	3.240	80	3.130	3.521	120	3.267	3.662

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Symbols and Terms

$\int_{+\infty}^{-\infty}$	= Integral from minus infinite to plus infinite
$\sqrt{}$	= Radical sign
$\sum$	= Sum symbol
$\prod$	= Product symbol
$\leq$	= Smaller than or equal to
$\geq$	= Greater than or equal to
$\neq$	= Not equal
$\lvert x \rvert$	= Absolute value of x (positive value of x)
B_{lower}, B_{upper}	= Factors for the calculation of limits of random variation of s
D	= Difference between the averages of two measurement series
d	= Difference between two measurement values
e	= Excess of a sample
e	= Base of the natural logarithm
f	= Number of degrees of freedom
F	= Quotient of the variances of two measurement series (Test statistic in F Test)
f(x)	= Probability density function
g	= Skewness of a sample
g_W	= Kurtosis of a sample
G_j	= Cumulative Frequency
h_j	= Relative frequency
$h_j(I)$, $h_j(II)$	= Relative frequency of partial collectives
$h'_j(I)$, $h'_j(II)$	= Non-standardized relative frequency of partial collectives
H_j	= Cumulative relative frequency
H_{lower}, H_{upper}	= Confidence limits used in the normal probability plot
H_0	= Null hypothesis
H_1	= Alternative hypothesis
i, j	= Count indices
k	= Class number or acceptance factor

k_A = Outlier factor

ln = Natural logarithm

m = Alternate sample size

m_k = kth moment of a sample

n = Sample size

n_j = Absolute frequency (Occupation number)

P = Probability

P_A = Probability (Confidence level)

$P(x_i)$ = Probability for characteristic value x_i

R = Range

r_{xy} = Correlation coefficient (estimated value)

s = Standard deviation of a sample

s_d = Standard deviation of a difference distribution of individual values

s_D = Standard deviation of the distribution of differences of averages

s_M = Standard deviation of the measurement error

s_{xy} = Covariance

s^2 = Variance of a sample

t = Factor for the calculation of the confidence interval for μ with unknown standard deviation of the population

u = Standardized random variable of the normal distribution $N(\mu = 0;\ \sigma^2 = 1)$

v = Coefficient of variation

x = Continuous characteristic values

x_i, y_i = Values of a measurement series

$x_{(1)}, \ldots, x_{(n)}$ = Values of a measurement series arranged in order of magnitude

$\overline{x}_g$ = Geometric mean of a sample

x_{max} = Largest value of a sample

x_{min} = Smallest value of a sample

$\tilde{x}$ = Median of a sample

$\overline{x}$ = Arithmetic mean of a sample

$\overline{x}_D$ = Average of the difference distribution from samples

α = Probability for type I error

β	= Probability for type II error
τ	= Skewness of a population
Δ	= Difference in the determination of the measurement accuracy
ε	= Shape parameter of the lognormal distribution
e	= Excess of a population
μ	= Average of a population
μ_D	= Average of the difference distribution
μ_g	= Geometric mean of a population
σ	= Standard deviation of a population
σ^2	= Variance of a population
π	= Number Pi (3.1416)

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